**Exercise: One-particle Bisognano–Wichmann property**

For sake of concreteness, let $\mathcal{H} = L^2(H_m, d\mu)$, where $m > 0$ and $H_m = \{p = (p_0, p_1, p_2, p_3) \in \mathbb{R}^{1+3} : p^2 = (p_0)^2 - (\vec{p})^2 = m^2, p_0 = 0\}$ be the one-particle space of a scalar (spin=0) massive particle with mass $m$, here:

$$\int f d\mu = \int_{\mathbb{R}^3} f(\omega_{\vec{p}}, \vec{p}) \frac{d\vec{p}}{\omega_{\vec{p}}} \equiv \int_{\mathbb{R}^{1+3}} f(p)\delta(p^2 - m^2)\theta(p_0) d^4 p\,, \quad \omega_p = \sqrt{m^2 + |\vec{p}|^2}\,.$$

(Indeed, you can take any Lorentz invariant measure on a Lorentz invariant subset of $\overline{V_+} \setminus \{0\}$).

Let $E\colon \mathcal{S}(\mathbb{R}^{1+3}, \mathbb{R}) \to \mathcal{H})$ be the restriction of the Fourier transformation to the mass shell:

$$Ef(p) = \int_{\mathbb{R}^{1+3}} \mathrm{e}^{\mathrm{i}p.x} f(x) dx, \qquad p \in H_m\,.$$

We remember that $\mathcal{P}_+^\uparrow$ acts unitarily on $\mathcal{H}$ by

$$[U(\Lambda, a)\xi](p) = \mathrm{e}^{\mathrm{i}p.a}\xi(\Lambda^{-1}p)\,, \quad (\Lambda, a) \in \mathsf{SO}(1,3)_0 \ltimes \mathbb{R}^{1+3} \equiv \mathcal{P}_+^\uparrow$$

and that $U(\Lambda, a)Ef = Ef_{(\Lambda,a)}$, where $f_{(\Lambda,a)}(x) = f(\Lambda^{-1}(x - a))$.

Prove that the $\beta = 1$ KMS condition holds for $H = E\mathcal{S}(W, \mathbb{R})$, where $W = \{x \in \mathbb{R}^{1+3}, |x_0| < x_1\}$ is the Rindler wedge and $V_t := U(\delta_{-t})$ and

$$\delta_t = \begin{pmatrix} \cosh(-2\pi t) & \sinh(-2\pi t) & 0 & 0 \\ \sinh(-2\pi t) & \cosh(-2\pi t) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \in \mathcal{L}_+^\uparrow \equiv \mathsf{SO}(1,3)_0\,.$$

Namely,

(1) $H$ is a real subspace and $H + \mathrm{i}H$ is dense in $\mathcal{H}$ (see Reeh–Schlieder property).

(2) $V_t H = H$.

(3) For every $\xi, \eta \in H$, there is a continuous bounded function $F\colon \mathbb{R} + \mathrm{i}[0,1] \to \mathbb{C}$ analytic on the interior, such that

$$F(t) = (\xi, V_t\eta)\,, \quad F(t + \mathrm{i}) = \overline{F(t)} \equiv (V_t\eta, \xi)$$

Or alternatively, for every $\xi \in H$, the $\mathcal{H}$ valued function $f(t) = V_t\xi$ extends to a bounded function on $\mathbb{R} + \mathrm{i}[0, 1/2]$ analytic on the interior, such that $f(t + \mathrm{i}/2) = Jf$ for the anti-unitary involution:

$$(J\xi)(p) = \overline{\xi(-jp)}$$

with $j \in \mathcal{L}_+$ given by $j\colon (p_0, p_1, p_2, p_3) \mapsto (-p_0, -p_1, p_2, p_3)$ and that $JV_tJ = V_t$ holds.

By the KMS uniqueness it follows that $\bar{H}$ is standard with modular operators ($\Delta^{\mathrm{i}t} = U(\delta_t), J$). In particular, the modular operators have geometric meaning. This is the so-called Bisognano–Wichmann property.

For a region $O \subset \mathbb{R}^{1+3}$ define $O'$ to be the union of all regions in $\mathbb{R}^{1+3}$ spacelike to $O$. We get a net $O \mapsto H(O)$ of standard subspaces, where $O$ is an causally-complete (i.e. $O'' = O$)

convex region. The net is defined by

$$H(O) = \bigcap_{g \in \mathcal{P}_+^\uparrow : O \subset gW} U(g)\bar{H}$$

Show that the net fulfills:

**Isotony:** If $O_1 \subset O_2$, then $H(O_1) \subset H(O_2)$.
**Locality:** If $O_1 \subset O_2'$, then $H(O_1) \subset H(O_2)'$.
**Covariance:** $U(g)H(O) = H(gO)$ for $g \in \mathcal{P}_+^\uparrow$ (for $g \in \mathcal{P}_+$).
**Reeh-Schlieder property:** $H(O)$ is standard.
**Factoriality:** $H$ is a factor, i.e. $H(O) \cap H(O)' = \{0\}$.

You don't need to use locality (i.e. $\mathrm{Im}(Ef, Eg) = 0$ if $\operatorname{supp} f$ is spacelike separated from $\operatorname{supp} g$).