1. Fermi Nets

Let $\mathcal{I}$ be the set of proper intervals $I \subset \mathbb{S}^1$, i.e. open, connected, non-empty, non-dense intervals.

Definition 1. A **graded local Möbius covariant net** or **Fermi net** $\mathcal{F}$ on $\mathbb{S}^1$ is a family $\{\mathcal{F}(I)\}_{I\in\mathcal{I}}$ of von Neumann algebras on a $\mathbb{Z}_2$-graded Hilbert space $\mathcal{H}$, i.e. there is a unitary operator Γ with $\Gamma^2 = 1$ with the following properties:

1. **Isotony.** $I_1 \subset I_2$ implies $\mathcal{F}(I_1) \subset \mathcal{F}(I_2)$.
2. **Graded locality.** $I_1 \cap I_2 = \emptyset$ implies $[\mathcal{F}(I_1), Z\mathcal{F}(I_2)Z^*] = \{0\}$, where $Z = \frac{1-\mathrm{i}\Gamma}{1-\mathrm{i}}$.
3. **Möbius covariance.** There is a unitary representation U of $\mathsf{M\ddot{o}b}^{(2)}$ on $\mathcal{H}$ such that $U(g)\mathcal{F}(I)U(g)^* = \mathcal{F}(gI)$.
4. **Positivity of energy.** U is a positive energy representation, i.e. the generator L_0 (conformal Hamiltonian) of the rotation subgroup $U(R(\theta)) = \mathrm{e}^{\mathrm{i}\theta L_0}$ has positive spectrum.
5. **Vacuum.** There is a (up to phase) unique rotation invariant and even (i.e. $\Gamma\Omega = \Omega$) unit vector $\Omega \in \mathcal{H}$ which is cyclic for the von Neumann algebra $\bigvee_{I\in\mathcal{I}} \mathcal{F}(I)$.

If $Z = 1$ then the net is called a **local Möbius covariant net** or **conformal net** Among the consequences of the definition are ([CKL08]):

Proposition 2. *Let $\mathcal{F}$ be a Fermi net on $\mathcal{H}$. Then the following holds.*

1. ***Reeh–Schlieder property.*** *For each $I \in \mathcal{I}$ the vector Ω is cyclic and separating for each $\mathcal{F}(I)$.*
2. ***Additivity.*** *If $I = \bigcup_i I_i$, then $\mathcal{F}(I) = \bigvee_i \mathcal{F}(I_i)$.*
3. ***Bisognano–Wichmann property.*** *The modular group $\Delta^{\mathrm{i}t}_{(\mathcal{F}(I),\Omega)}$ is given by $U(\delta_I(-2\pi t))$.*
4. ***Twisted Haag duality*** *or* ***Haag–Araki duality.*** *For $I \in \mathcal{I}$ we have $\mathcal{F}(I)' = Z\mathcal{F}(I')Z^*$.*
5. ***Irreducibility.*** *We have $\bigvee_{I\in\mathcal{I}} \mathcal{F}(I) = \mathrm{B}(\mathcal{H})$.*

2. Examples

Free Bosonic net ($\mathsf{U}(1)$–current net). Let U_1 be the irreducible lowest weight $\ell = 1$ representation of $\mathsf{Möb}$ on $\mathcal{H}_1 \cong \overline{C^\infty(S^1,\mathbb{R})/\mathbb{R}}^{\|\cdot\|}$. Then using Bosonic second quantization we define a conformal net:

$$\mathcal{A}(I) := \{W(f) : \operatorname{supp} f \subset I\}'' \subset \mathrm{B}(\mathcal{H}_\mathcal{A}), \qquad \mathcal{H}_\mathcal{A} := \Gamma(\mathcal{H}_1),$$
$$U_\mathcal{A}(g) := \Gamma(U_1(g)), \qquad \Omega_\mathcal{A} = e^0.$$

The Fourier modes $\{j_m\}_{m\in\mathbb{Z}}$ informally given by $W(f) = \exp(\mathrm{i} \sum \hat{f}_k j_k)$ fulfill:

$$[j_n, j_m] = m\delta_{m+n,0}$$
$$j_m\Omega_\mathcal{A} = 0 \Leftrightarrow m = 0, 1, 2, \ldots$$
$$U(R(\theta)) j_m U(R(\theta))^* = \mathrm{e}^{-\mathrm{i}m\theta} j_m$$

Complex free Fermi net. Let $\mathsf{U}_\mathbb{C}$ be the lowest weight representation of with lowest weight $\ell = \frac{1}{2}$ and multiplicty 2 on $\mathcal{H}_\mathbb{C} \cong \overline{L^2(S^1,\mathbb{C})} \oplus P^\perp L^2(S^1)$. We get a Fermi net:

$$\mathcal{F}(I) := \{c(f) : \operatorname{supp} f \subset I\}'' \subset \mathrm{B}(\mathcal{H}_\mathcal{F}), \qquad \mathcal{H}_\mathcal{F} := \Lambda(\mathcal{H}_\mathbb{C}),$$
$$U_\mathcal{F}(g) := \Lambda(U_\mathbb{C}(g)), \qquad \Omega_\mathcal{F} = 1 \in \Lambda^0(\mathcal{H}_\mathbb{C}),$$
$$V(\vartheta) = \Lambda(\mathrm{e}^{-\mathrm{i}\vartheta}).$$

The Fourier modes $\{\psi_r, \bar\psi_r\}_{r\in\frac{1}{2}+\mathbb{Z}}$ with $c(e_n) = \psi_{-n-\frac{1}{2}} + \bar\psi_{n+\frac{1}{2}}$ fulfill:

$$\{\psi_r, \psi_s\} = \{\bar\psi_r, \bar\psi_s\} = 0, \qquad \{\psi_r, \bar\psi_s\} = \delta_{r+s}, 0$$
$$\psi_r\Omega_\mathcal{F} = \bar\psi_r\Omega_\mathcal{F} = 0 \Leftrightarrow m = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \ldots$$
$$U(R(\theta))\psi_r U(R(\theta))^* = \mathrm{e}^{-\mathrm{i}r\theta}\psi_r, \qquad U(R(\theta))\bar\psi_r U(R(\theta))^* = \mathrm{e}^{-\mathrm{i}r\theta}\bar\psi_r,$$
$$V(\vartheta)\psi_r V(\vartheta))^* = \mathrm{e}^{\mathrm{i}\vartheta}\psi_r, \qquad V(\vartheta)\bar\psi_r V(\vartheta))^* = \mathrm{e}^{\mathrm{i}\vartheta}\bar\psi_r.$$

References

[CKL08] S. Carpi, Y. Kawahigashi, and R. Longo, *Structure and classification of superconformal nets*, Ann. Henri Poincaré **9** (2008), no. 6, 1069–1121. MR2453256 (2009j:81082)