LECTURE NOTES: ALGEBRAIC AND TOPOLOGICAL QUANTUM FIELD THEORY
CHAPTER 1: QUANTUM MECHANICS

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ABSTRACT. ATTENTION: not proof read lecture notes in progress.....

CONTENTS

Quantum mechanics (QM) was born in two different ways to explain the quantization of energy levels in atoms:

- Heisenberg: He replaced the variables q, p with matrices double indexed quantities q_{nm}, p_{nm}, where n, m describe energy elvels and x_{nm} the intensity of sprectral line for between level n and m. Born and Jordan realized that this was calculation with infinite matrices
- Schrödinger: He tried to understand quantization (of spectral lines) as an eigevalue problem. Inspired by the idea of wave mechanics of deBrouglie, he replaced the Hamiltonian with a differential operator.

Through work of Schrödinger, Dirac, Jordan and von Neumann it became clear that these two approaches are mathematical equivalent.

Abstractly, in a Quantum Theory we start with a Hilbert space $\mathcal{H}$. This is a vector space with $\mathcal{H}$ a positive definite sesquilinar form $(\,\cdot\,,\,\cdot\,)\colon \mathcal{H}\times\mathcal{H}\to\mathbb{C}$:

(1) $(x,y)=\overline{(y,z)}$ for all $x,y\in\mathcal{H}$
(2) $x\mapsto(y,x)$ is a $\mathbb{C}$-linear map.
(3) $(x,x)>0$ for $x\neq 0$.

which is a complete space with respect to the norm $\|x\| = \sqrt{(x,x)}$.

We associate:

$$\text{measurable quantities in classical mechanics} \longmapsto \text{self-adjoint operators}$$

Date: today.

the spectral values of the operator are the possible values of the measurement of the quantity in question. The concrete description of the Hilbert space and or a possible choice of a basis are irrelavent. Important is the characterization of observables.

Two systems of obserables on $\mathcal{H}_1$ and $\mathcal{H}_2$ are equivalent if there exists a unitary $U\colon \mathcal{H}_1 \to \mathcal{H}_2$, such that

$$A_2 = UA_1U^*$$

for every observable A, where A_1 and A_2 are the representation of the observable A on $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively.

Strategy for quantum mechanics.

1.1. Classical mechanics

In classical mechanics the trajectory of a particle $q(t) = (q_1(t), \ldots, q_n(t))$ depending on the time t is obtained by solving the initial value problem given by Newton's equation:

$$m_i\ddot{q}_i(t) = F_i(q(t))\,, \qquad F_i(q) = -\frac{\partial V(q)}{\partial q_i}\,, \qquad \dot{x}(q) = \frac{\mathrm{d}x(t)}{\mathrm{d}t}\,, \tag{1.1}$$

where $F = (F_i)$ is the force and V is a potential.

To go to quantum mechanics we pass to Hamiltonian mechanics. One introduces so-called **canonical coordinates**, which are positions $q = (q_i)$ and conjugate momenta $p = (p_i)$, where $i = 1, \ldots, n$ runs over the degrees of freedom. The time $t \in \mathbb{R}$ is an external parameter. **Observables** are smooth functions of these variables, i.e. functions on so-called **configuration space** which we take for simplicity to be $\{(q_1, \ldots, q_n, p_1, \ldots, p_n) \in \mathbb{R}^2 n\}$. They also might depend on t, but most of the time we will assume they do not. Then one introduces the Poisson bracket:

$$\{f, g\} = \sum_{i=1}^{n} \left(\frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q_i} \right)$$

In particular, we have the relations:

$$\{q_i, q_j\} = 0\,, \tag{1.2}$$

$$\{p_i, p_j\} = 0\,, \tag{1.3}$$

$$\{q_i, p_i\} = \delta_{ij}\,. \tag{1.4}$$

Dynamics of the system is encoded via the Hamiltion H, which is itself an observable corresponding to the **energy** given by

$$H(p, q) = \sum_i \frac{p_i^2}{2m_i} + V(q)\,.$$

which for simplicity is just the sum of kinetic and potential energy: Then differential equation

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \{f, H\} + \frac{\partial f}{\partial t}$$

determines how observables evolve in time. In particular, we get:

$$\begin{cases} \dot{q}_i = \{q, H\} = \dfrac{\partial H}{\partial p_i} = \dfrac{p_i}{m_i} \\ \dot{p}_i = \{p, H\} = -\dfrac{\partial H}{\partial q_i} = -\partial V(q)\partial q_i \end{cases} \tag{1.5}$$

so this formalism is indeed equivalent to Newton's equation.

1.2. Quantum mechanics

So in quantum mechanics we have to replace (q_i), (p_i) with operators (Q_i), (P_i), such that

$$[Q_i, Q_j] = 0\,, \tag{1.6}$$
$$[P_i, P_j] = 0\,, \tag{1.7}$$
$$[Q_i, P_i] \supset \mathrm{i}\hbar\delta_{ij}\,, \tag{1.8}$$

where $[\,\cdot\,,\,\cdot\,]$ is the commutator:

$$[x, y] = xy - yx\,.$$

The idea of the so-called **canonical quantization** is to replace p, q with non-commuting operators P, Q and the Poisson bracket with the commutator:

$$\mathrm{i}\hbar\{\,\cdot\,,\,\cdot\,\} \longrightarrow [\,\cdot\,,\,\cdot\,]\,. \tag{1.9}$$

We will assume that we are using units such that $\hbar = 1$. The operators are necessary unbounded.

We will make use of the following result. Stone von Neumann theorem.

Theorem 1.2.1. *Let $\mathcal{H}$ be a Hilbert space. Then there is a one-to-one correspondence between:*

- *(Possible unbounded) self-adjoint operators P on $\mathcal{H}$ and*
- *strongly continuous one parameter groups $(U_t)_{t\in\mathbb{R}}$ of unitaries.*

given by $P \mapsto (U_{P,t})_{t\in\mathbb{R}}$ with $U_{P,t} := \mathrm{e}^{\mathrm{i}tP}$ using the spectral calculus.

Example 1.2.2 (Schrödinger representation)**.** For simplcity let us consider $n = 1$. Let $\mathcal{H} = L^2(\mathbb{R})$. Then there are self-adjoint operators P, Q defined by the one-parameter groups:

$$(\mathrm{e}^{\mathrm{i}tQ} f)(q) = \mathrm{e}^{\mathrm{i}tq} f(q) \tag{1.10}$$
$$(\mathrm{e}^{\mathrm{i}tP} f)(p) = f(q + t) \tag{1.11}$$

On a dense domain Q is the multiplication operator $f(q) \mapsto qf(q)$. Using Fourier transformation:

$$\hat{f}(p) = (2\pi)^{-\frac{n}{2}} \int\limits_{\mathbb{R}^n} \mathrm{e}^{-\mathrm{i}px} f(x)\,\mathrm{d}^n x \tag{1.12}$$
$$f(x) = (2\pi)^{-\frac{n}{2}} \int\limits_{\mathbb{R}^n} \mathrm{e}^{\mathrm{i}px} \hat{f}(p)\,\mathrm{d}^n p \tag{1.13}$$

with $(\hat{f}, \hat{g}) = (f, g)$ (Plancharel theorem).

$$f(q + t) = (2\pi)^{-\frac{n}{2}} \int\limits_{\mathbb{R}^n} \mathrm{e}^{\mathrm{i}p(q+t)} \hat{f}(p)\,\mathrm{d}^n p \tag{1.14}$$
$$= (2\pi)^{-\frac{n}{2}} \int\limits_{\mathbb{R}^n} \mathrm{e}^{\mathrm{i}pt}\mathrm{e}^{\mathrm{i}pq} \hat{f}(p)\,\mathrm{d}^n p \tag{1.15}$$
$$= (2\pi)^{-\frac{n}{2}} \mathrm{e}^{t\partial_q} \int\limits_{\mathbb{R}^n} \mathrm{e}^{\mathrm{i}pq} \hat{f}(p)\,\mathrm{d}^n p \tag{1.16}$$
$$= \mathrm{e}^{\mathrm{i}t(-\mathrm{i}\partial_q)} f(q) \tag{1.17}$$

we get: P is $-\mathrm{i}\partial_q$. In particular, $[P, Q] \supset \mathrm{i} := \mathrm{i}\cdot 1$.

If we replace $\mathrm{i}\hbar\{\,\cdot\,,\,\cdot\,\} \leftrightarrow [\,\cdot\,,\,\cdot\,]$ (correspondence principle) in the Hamilton equation we get the **Heisenberg equation**

$$\frac{\mathrm{d}}{\mathrm{d}t} A = \mathrm{i}[H, A]$$

To make sense we need that H is a self-adjoint operator. Then

$$U(t) = \mathrm{e}^{\mathrm{i}tH}$$

$$A_{t+s} = U(t)A_sU(t)^*$$

One seperates the physical system between observables and states. This is sometimes called the **Heisenberg cut:**

- Observables describe properties of measurement devices
- States describe properties of prepared ensembles

In the **Hilbert space formalism** with Hilbert space $\mathcal{H}$:

- **Observables:** Self-adjoint operators on $\mathcal{H}$
- **States:** density matrices ρ on $\mathcal{H}$, i.e. $\rho \geq 0$, $\operatorname{tr}\rho = 1$.

Definition 1.2.3. Let $\mathcal{H}$ be a Hilbert space and $\{e_i\}$ be a (countable) normal basis. We define the trace to be

$$\operatorname{tr}(A) = \sum_i (e_i, Ae_i)\,. \tag{1.18}$$

Definition 1.2.4. Let $\mathcal{H}$ be a Hilbert space. For $a, b \in \mathcal{H}$ we denote by $|a\rangle\langle b|$ the **rank one operator** given by $|a\rangle\langle b|\, x = (b, x) \cdot a$.

It holds

- $|a\rangle\langle b|^* = |b\rangle\langle a|$
- $|a\rangle\langle b|$ is a projection if and only if $a = b$ with $\|a\| = 1$.

Pure states are given by unit rays $[\psi] = \{c\psi : c \in \mathbb{T}\}$ with $\psi = 1$. Equivalently, they are given by density matrices with $\rho^2 = \rho$. In this case, $\rho = |\psi\rangle\langle\psi|$ for some eigenvector $\psi \in \mathcal{H}$ with $\|\psi\| = 1$ and

$$\operatorname{tr}(\rho A) = \sum_i (e_i, |\psi\rangle\langle\psi|\, e_i) = (\psi, |\psi\rangle\langle\psi|\, \psi) = (\psi, A\psi) \tag{1.19}$$

With $A := A_0$ and the Heisenberg equation we get the time evolution:

$$\operatorname{tr}(\rho A_t) = \operatorname{tr}(\rho U(t)AU(t)^*) = \operatorname{tr}(U(t)^*\rho U(t)A) \tag{1.20}$$

$$\rho_{s+t} = U(t)\rho_s U(-t)^* \tag{1.21}$$

$$\tag{1.22}$$

the von Neumann equation:

$$\mathrm{i}\hbar\frac{\mathrm{d}}{\mathrm{d}t}\rho = [H, \rho] \tag{1.23}$$

For the special case of pure state $\rho = |\psi\rangle\langle\psi|$ we get the Schrödinger equation:

$$\mathrm{i}\hbar\frac{\mathrm{d}}{\mathrm{d}t}\psi = H\psi\,. \tag{1.24}$$

- **Heisenberg picture:** The observables are time dependent and evolve with the Heisenberg equation, and the states are fixed.
- **Schrödinger equation:** The states are time dependent and evolve via the Schrödinger equation and the observables are time-independent.

Definition 1.2.5. The free polynomial $*$-algebra $\mathcal{P}$ is a complex vector space with basis words in $\{Q_j, P_k\}_{k,j=1,\dots,n}$, with product

$$(Q_{j_1}^{m_1} P_{k_1}^{n_1} \cdots Q_{j_i}^{m_i} P_{k_i}^{n_i})(Q_{j_{i+1}}^{m_{i+1}} P_{k_{i+1}}^{n_{i+1}} \cdots Q_{j_\ell}^{m_\ell} P_{k_\ell}^{n_\ell}) = Q_{j_1}^{m_1} P_{k_1}^{n_1} \cdots Q_{j_i}^{m_i} P_{k_i}^{n_i} Q_{j_{i+1}}^{m_{i+1}} P_{k_{i+1}}^{n_{i+1}} \cdots Q_{j_\ell}^{m_\ell} P_{k_\ell}^{n_\ell} \tag{1.25}$$

and adjoints $Q_j^* = Q_j$, $P_k^* = P_k$,

$$\sum c_{j_1,k_1,\dots,j_i,k_i}^{m_1,n_1,\dots,m_i,n_i} (Q_{j_1}^{m_1} P_{k_1}^{n_1} \cdots Q_{j_i}^{m_i} P_{k_i}^{n_i})^* = \sum \overline{c_{j_1,k_1,\dots,j_i,k_i}^{m_1,n_1,\dots,m_i,n_i}} P_{k_i}^{n_i} Q_{j_i}^{m_i} \cdots P_{k_1}^{n_1} Q_{j_1}^{m_1} \tag{1.26}$$

and unit 1 (corresponding to the empty word).

Definition 1.2.6. The Heisenberg algebra is the quotient $\mathcal{P}/\mathcal{I}$ by the two sided $*$-ideal $\mathcal{I}$ generated by:

$$A(Q_iQ_j - Q_jQ_i)B, \qquad A(P_iP_j - P_jP_i)B, \qquad A(Q_iP_j - P_jQ_i - \mathrm{i}\delta_{i,j}1)B, \tag{1.27}$$

for all $A, B \in \mathcal{P}$.

1.3. The Weyl algebra

The elements of the Heisenberg algebra are intrinsically unbounded.

One way to avoid is to take bounded functions.

First observe the Heisenberg algebra is generated by symbols $\sum_k u_k P_+ v_k Q_k$ and products of them.

We use $z = u + \mathrm{i}v \in V := \mathbb{C}^n$ and denote $H(z) = \sum_k u_k P_+ v_k Q_k$ and we get

$$[H(z), H(w)] = -\mathrm{i}\omega(z, w) \qquad \omega(z, z') = \mathrm{Im}(z, z') = u \cdot v' - v' \cdot u \tag{1.28}$$

(V, ω) is a symplectic vector space.

Our goal is to take $W(z) = \exp(\mathrm{i}H(z))$, but we cannot do this directly since exp is not defined on symbols P_i, Q_i, since we just have finite direct sums of words in it.

Instead we consider the abstract symbols $W(z)$ satisfying the expected relations, which we want to derive now. If $[A, [A, B]] = [B, [A, B]] = 0$ one gets the Baker-Campbell-Hausdorff formula

$$e^A e^B = e^{\frac{1}{2}[A,B]} e^{A+B} \tag{1.29}$$

we get formally

$$W(z)W(z') = \mathrm{e}^{\frac{\mathrm{i}}{2}\omega(z,z')} W(z + z') \,. \tag{1.30}$$

This motivates the following definition.

Definition 1.3.1. The **(pre-) Weyl algebra** $\mathcal{W}$ is the free polynomial $*$-algebra generated by symbols $W(z), W(z)^*$ with $z \in V$ modulo the two sided $*$-ideal generated by:

$$W(z)W(z') - \mathrm{e}^{\frac{\mathrm{i}}{2}\omega} W(z + z') \qquad W(z)^* - W(-z) \tag{1.31}$$

Exercise: We have the following properties:

(1) $W(0) = 1$ (by uniqueness of unity)
(2) **Weyl operators are unitaries**, i.e. $W(z)W(z)^* = W(z)^*W(z) = 1$.
(3) Every element in $\mathcal{W}$ is of the form $cW(z)$ with $c \in \mathbb{C}$ and $z \in V$.

Remark 1.3.2. The group generated by symbols $\{cW(z) : c \in \mathbb{T}, z \in V\}$ with the same relations is sometimes called Heisenberg group, so the Weyl algebra is the group algebra of the Heisenberg group. We will come back to this if we study central extensions.

1.4. Representations of the Weyl algebra

We are interested in the representation theory of the Weyl algebra.

Definition 1.4.1. A $*$-representation is a unital $*$-homomorphism $\pi\colon \mathcal{W} \to \mathrm{B}(\mathcal{H})$, i.e. a linear map fulfilling:

(1) **multiplicativity:** $\pi(w_1)\pi(w_2) = \pi(w_1 w_2)$,
(2) **$*$-property:** $\pi(w^*) = \pi(w)^*$
(3) **unital:** $\pi(1) = 1 \equiv 1_{\mathcal{H}}$.

The Schrödinger representation gives a representation of the Weyl algebra, heuristically using the Baker-Campbell-Hausdorff:

$$\left(\mathrm{e}^{\mathrm{i}(uP+qv)} f\right)(q) = \mathrm{e}^{\frac{\mathrm{i}}{2}uv}\left(\mathrm{e}^{\mathrm{i}vQ}\mathrm{e}^{\mathrm{i}uP} f\right)(x) \tag{1.32}$$

$$= \mathrm{e}^{\frac{\mathrm{i}}{2}uv}\mathrm{e}^{\mathrm{i}vq} f(q+u) \tag{1.33}$$

Example 1.4.2. Let $\mathcal{H} = L^2(\mathbb{R})$ and define $\pi_{1,2}\colon \mathcal{W} \to \mathcal{B}(\mathcal{H})$ by

$$(\pi_1(W(u+\mathrm{i}v))f(q) = \mathrm{e}^{\frac{\mathrm{i}}{2}uv}\mathrm{e}^{\mathrm{i}vq}(q+u) \tag{1.34}$$

$$(\pi_2(W(u+\mathrm{i}v))f(p) = \mathrm{e}^{-\frac{\mathrm{i}}{2}uv}\mathrm{e}^{\mathrm{i}up}(p-v) \tag{1.35}$$

π_1 and π_2 are called the **Schrödinger representation** of the Weyl algebra in position and momentum space, resepectively.

Definition 1.4.3. Let $\pi_1 = (\pi_1, \mathcal{H}_1)$ and $\pi_2(\pi_2, \mathcal{H}_2)$ be two representations. Then

$$\mathrm{Hom}_{\mathcal{W}}(\pi_1, \pi_2) := \{T \in \mathrm{B}(\mathcal{H}_1, \mathcal{H}_2) : T\pi_1(\,\cdot\,) = \pi_2 T\} \tag{1.36}$$

They are called unitary equivalent if there is a unitary $U\colon \mathcal{H}_1 \to \mathcal{H}_2$ in $\mathrm{Hom}_{\mathcal{W}}(\pi_1, \pi_2)$.

Example 1.4.4. Let $\mathcal{F}\colon \mathcal{H} \to \mathcal{H}$ be the unitary Fourier transform $\mathcal{F}f := \hat{f}$. Then

$$\pi_2(W) = \mathcal{F}\pi_1(W)\mathcal{F}^{-1}\,,\ \text{i.e. } \mathcal{F} \in \mathrm{Hom}_{\mathcal{W}}(\pi_1, \pi_2) \tag{1.37}$$

thus the Schrödinger representations of the Weyl algebra in position and momentum space are unitarily equivalent.

Definition 1.4.5. Let $\pi = (\pi, \mathcal{H})$ be a representation of $\mathcal{W}$. A closed space $\mathcal{K} \subset \mathcal{H}$ is called inariant under $\mathcal{W}$ if $\pi(W)\mathcal{K} \subset \mathcal{K}$. We say that π is irreducible if the only invariant subspaces are $\mathcal{H}$ and $\{0\}$.

Lemma 1.4.6 (Schur's Lemma)**.** *Let $\pi = (\pi, \mathcal{H})$ be a representation of $\mathcal{W}$. The following are equivalent:*

(1) π is irreducible,
(2) $\overline{\pi(\mathcal{W})\xi} = \mathcal{H}$ for every $\xi \in \mathcal{H} \setminus \{0\}$,
(3) $\mathrm{Hom}_{\mathcal{W}}(\pi, \pi) = \mathbb{C} \cdot 1$,
(4) $\pi(\mathcal{W})' =: \{x \in \mathrm{B}(\mathcal{H}) : [x, \pi(\mathcal{W})] = \{0\}\} = \mathbb{C} \cdot 1$.

Exercise: The Schrödinger representation is irreducible.

Remark 1.4.7. The representations of $\mathcal{W}$ form a category $\mathrm{Rep}(\mathcal{W})$ with:

Objects: Representations $\pi = (\pi, \mathcal{H})$ of $\mathcal{W}$.
Morphisms: $\mathrm{Hom}_{\mathcal{W}}(\pi, \pi)$.

$\mathrm{Rep}(\mathcal{W})$ is actually a C^*-category.

We have a more unphysical example of a representation of $\mathcal{W}$.

Example 1.4.8. Let $\mathcal{H}_3 = \ell^2(\mathbb{R}^n)$ with ONB $\{e_p\}_{p\in\mathbb{R}_n}$

$$\pi_3(W(u+\mathrm{i}v))e_p = \mathrm{e}^{-\frac{\mathrm{i}}{2}uv}\mathrm{e}^{\mathrm{i}up}e_{p+v} \tag{1.38}$$

π_3 is irreducible. But π_3 is not equivalent to $\pi_1 \cong \pi_2$, because $\mathcal{H}_3$ is non-separable and there is not unitary between a seperable and non-seperable Hilbert space.

The physical interesting representations should fulfill the **criterion**:

$$z \longmapsto (\xi, W(z)\xi) \tag{1.39}$$

is continuous, i.e. in any state the observables depend continuously on the z. By Stone's theorem it follows that

$$\pi(W(u)) = \exp\left(\mathrm{i}\sum_i u_i P_i^\pi\right) \tag{1.40}$$

$$\pi(W(\mathrm{i}v)) = \exp\left(\mathrm{i}\sum_i v_i Q_i^\pi\right) \tag{1.41}$$

where $\{P_i^\pi\}$ and $\{Q_i^\pi\}$ are two families of pairwise commuting of self-adjoint operators. By taking a derivative of the Weyl relations one gets $[Q_i^\pi, P_j^\pi] \supset \mathrm{i}\delta_{i,j}$.

Theorem 1.4.9 (Stone–von Neumann REF)**.** *Let $0 \leq n < \infty$. Then every irreducible representation π of $\mathcal{W}$ is unitarily equivalent to the Schrödinger representationt $\pi_{1,2}$.*

This theorem does not generalize to the case of infinitely many degrees of freedom which will find in QFT. Symmetries, which we will see are prescribed by groups, will play a prominent role to distinguish between these different representations.

1.5. States

Definition 1.5.1. If we have a representation π of $\mathcal{W}$ and a density matrix ρ ($\mathrm{B}(\mathcal{H}) \ni \rho \geq 0, \operatorname{tr}\rho = 1$) on $\mathcal{H}$, we can associate the expectation value

$$\mathcal{W} \ni W \longmapsto \langle W\rangle_{\pi,\rho} = \operatorname{tr}(\rho\pi(W))\,. \tag{1.42}$$

We call this a **physical state**.

Exercise: The expextation value $\varphi(\,\cdot\,) = \langle\,\cdot\,\rangle_{\pi,\rho}$ is a **state** on $\mathcal{W}$, namely it is a linear functional $\varphi\colon \mathcal{W} \to \mathbb{C}$, which is

(1) **normalized**, i.e. $\varphi(1) = 1$,
(2) **positive**, i.e. $\varphi(w^*w) \geq 0$ for all $w \in \mathcal{W}$.

Definition 1.5.2. Let $\pi = (\pi, \mathcal{H})$ be a representation. A vector $\Omega \in \mathcal{H}$ is called **cyclic** if $\overline{\pi(\mathcal{W})\Omega} = \mathcal{H}$. The representation π is called **cyclic** if it contains a cyclic vector $\Omega \in \mathcal{H}$ (Note this is a property). By abuse of terminology we will also call $(\pi, \mathcal{H}, \Omega)$ a **cyclic representation**.

Proposition 1.5.3 (Gelfand–Naimark–Segal construction)**.** *Let $\varphi\colon \mathcal{W} \to \mathbb{C}$ be a state. Then there exists a cyclic representation $(\pi_\varphi, \mathcal{H}_\varphi, \Omega_\varphi)$, such that:*

$$\varphi(x) = (\Omega_\varphi, \pi_\varphi(x)\Omega_\varphi) \equiv \operatorname{tr}\Big(|\Omega_\varphi\rangle\langle\Omega_\varphi|\, x\Big)\,. \tag{1.43}$$

Proof. This is the GNS construction. Take the pre-Hilbert space $\mathcal{H}_0 := \mathcal{W}$ with inner product $(x, y) := \varphi(x^*y)$. By the Cauchy–Schwarz inequality the vectors $\mathcal{I} = \{x \in \mathcal{H}_0 : \varphi(x^*x)\}$ with zero norm form a subspace. Then it can be shown that

$$\mathcal{H} := \overline{\mathcal{H}_0/\mathcal{I}}^{\|\cdot\|} \tag{1.44}$$

$$\pi(f) := [f] \tag{1.45}$$

$$\Omega := [1] \tag{1.46}$$

defines a representation and

$$(\Omega, \pi(x)\Omega) = ([1], \pi(x)[1]) = ([1], [x]) = \varphi(1^*x) = \varphi(x)\,. \tag{1.47}$$

□

Corollary 1.5.4. *Every state $\omega\colon \mathcal{W} \to \mathbb{C}$ on $\mathcal{W}$, i.e. every normalized positive functional is a physical state in the sense of Definition 1.5.1.*

Proposition 1.5.5. *If $\pi_1 \cong \pi_2$, then the corresponding set of physical states coincide.*

Proof. Let ρ_1 be a density matrix for π_1, then

$$\operatorname{tr}(\rho_1\pi_1(W)) = \operatorname{tr}(\rho_1 U\pi_2(W)U^*) = \operatorname{tr}(U^*\rho_1 U\pi_2(W)) \tag{1.48}$$

so it is given by the density matrix $\rho_2 = U^*\rho_1 U$ in π_2. □

One can definite the Weyl C* algebra.

Definition 1.5.6. We define a seminorm:

$$\|W\| := \sum_\pi \|\pi(W)\| \tag{1.49}$$

on $\mathcal{W}$, where the π runs over all cyclic representations. We denote by $\tilde{\mathcal{W}} := \overline{\mathcal{W} = \mathcal{W}/\mathcal{N}}^{\|\cdot\|}$ the completion of $\mathcal{W}$ ($\mathcal{N} = \{W \in \mathcal{W} : \|W\| = 0\}$).

- The supremum is finite, because we have for any representation π:

$$\|\pi(W(z)\|^2 = \|\pi(W(z))^*\pi(W(z)) \tag{1.50}$$

$$| = \|\pi(1)\| = 1\,. \tag{1.51}$$

- Using the GNS representation one can show that:

$$\|W\| := \sup_\varphi \varphi(W^*W)^{\frac{1}{2}} \tag{1.52}$$

(Exercise).
- We have the Banach algebra and C*-property $\|xy\| \leq \|x\|\|y\|$ and $\|xx^*\| = \|x\|^2$, respecitively.
- Indeed the norm is a seminorm.

Remark 1.5.7. A state is fixed by the value it takes on the Weyl unitaries $W(f)$.

1.6. Symmetries

Physical input: Symmetries are described by automorphisms of $\mathcal{W}$.

Definition 1.6.1. An automorphism is a bijective linear map $\alpha\colon\ : \mathcal{W} \to \mathcal{W}$ satisfying:

(1) $\alpha(x_1) = \alpha(x_1)\alpha(x_2)$,
(2) $\alpha(x^*) = \alpha(x)^*$,
(3) $\alpha(1) = \alpha$.

The **group of automorphisms** of $\mathcal{W}$ is denoted by $\mathrm{Aut}(\mathcal{W})$.

Definition 1.6.2. An automorphism is called **inner** if $\alpha(\,\cdot\,) = \mathrm{Ad}\,U := U^*(\,\cdot\,)U$ for some unitary $U \in \mathcal{W}$. The **group of inner automorphisms** of $\mathcal{W}$ is denoted by $\mathrm{Inn}(\mathcal{W})$.

Example 1.6.3. (1) For $U = W(u_0)$ with $u_0 \in \mathbb{R}^n$, we have a one-parameter group of automorphisms:

$$\alpha_{u_0}(W(u + \mathrm{i}v)) = W(u_0)W(u + \mathrm{i}v)W(u_0)^{-1} = \mathrm{e}^{\mathrm{i}u_0\cdot v}W(u + \mathrm{i}v) \tag{1.53}$$

which is the translation of coordinates in π_1

$$\pi_1(\alpha_{u_0}(W(z)) = \mathrm{e}^{\mathrm{i}u_0\cdot v}\mathrm{e}^{\mathrm{i}(uP+vQ)} = \mathrm{e}^{\mathrm{i}(uP+v(Q+u_0))}\,. \tag{1.54}$$

(2) Let $U = W(\mathrm{i}v_0)$ with $v_0 \in \mathbb{R}^n$, we have

$$\alpha_{v_0}(W(u + \mathrm{i}v)) = W(v_0)W(u + \mathrm{i}v)W(v_0)^{-1} - \mathrm{e}^{=\mathrm{i}v_0\cdot u}W(u + \mathrm{i}v) \tag{1.55}$$

which is a translation in momentum space.

Example 1.6.4. Let $R \in \mathsf{SO}(n)$. Then

$$\alpha_R(W(z)) = W(Rz) \tag{1.56}$$

is an automorphism which is not inner.

Example 1.6.5. Let $n = 3$ and R be the rotation around the z-axis by the angle θ. Then in the Schrödinger representation:

$$\pi_1(\alpha_R(W(z)) = U\pi_1(W(z))U^* \tag{1.57}$$

with $U = \mathrm{e}^{\mathrm{i}\theta L_3}$ and $L_3 = Q_1P_2 - Q_2P_1$.

Definition 1.6.6. Let $\pi = (\pi, \mathcal{H})$ be a representation of $\mathcal{W}$. Then $\alpha \in \mathrm{Aut}(\mathcal{W})$ is said to be **unitarily implementable** on $\mathcal{H}$, if there exists a unitar $U \in \mathrm{B}(\mathcal{H})$ (we then write also $U \in \mathsf{U}(\mathcal{H})$), such that

$$\pi(\alpha(w)) = U\pi(w)U^*\,, \qquad w \in \mathcal{W}\,. \tag{1.58}$$

Example 1.6.7. A large class of automorphisms are of the form:

$$\alpha(W(z)) = c(z)W(Sz) \tag{1.59}$$

The Weyl relations demand that

- $c\colon \mathbb{C}^n \to \mathbb{T}$ is a group representation (a character) and
- $S : \mathbb{C}^n \to \mathbb{C}^n$ is a real linear and a symplectic transformation, i.e. $\mathrm{Im}(Sz, Sz) = (z, z')$.

1.7. Excursion: Uniqueness of the Weyl C*-algebra

Let V be a real (finite-dimensional?) vector space and ω be a (non-degenerated) symplectic form. The Weyl algebra $\mathcal{W} := \mathcal{W}(V, \omega)$ is the free $*$-algebra with relations

$$W(x)W(y) = \mathrm{e}^{-\mathrm{i}\omega(x,y)}W(x + y), W(x)^* \qquad = W(-x) \tag{1.60}$$

Proposition 1.7.1. *The norm $\|\pi(w)\|$ for $w \in \mathcal{W}$ is independent of the representation.*

Proof. On $\ell^2(V)$ we get a representation:

$$[\pi_s(W(x))f](y) = \mathrm{e}^{\mathrm{i}\omega(x,y)}[R(x)f](y) = \mathrm{e}^{\mathrm{i}\omega(x,y)}f(x + y) \tag{1.61}$$

Let $\hat{V} = \{\chi\colon V \to \mathbb{T}\}$ the Pontryagin dual of V (seen as a discrete abelian group) and let $\mathcal{F}\colon \ell^2(V) \to L^2(\hat{V})$ be the unitary Fourier transformation. Then $\hat{R}(x) = \mathcal{F}R(x)\mathcal{F}^{-1} \in \mathsf{U}(L^2(V))$ is given by

$$[R(y)g](\chi) = \chi(y)g(\chi) \qquad g \in L^2(V), \chi \in \hat{V}\,. \tag{1.62}$$

Let $\pi\colon \mathcal{W} \to \mathrm{B}(\mathcal{H})$ be an abitrary (non-trivial) representation. On $\mathcal{H}\otimes\ell^2(V) = \ell^2(V,\mathcal{H})$ we introduce the unitary

$$U\psi(x) = \pi(W(x))\psi(x) \qquad \psi \in \ell^2(V,\mathcal{H}),\ z \in V\,. \tag{1.63}$$

Then

$$U\pi(W(x)) \otimes R(x) = 1 \otimes \pi_s(W(x)\,. \tag{1.64}$$

Then

$$\left\|\sum_i \lambda_i \pi_s(W(z_i))\right\| = \left\|\sum \lambda_i \pi(W(z_i)) \otimes R(z_i)\right\| \tag{1.65}$$

$$(\text{Fourier}) = \left\|\sum_i \lambda_i \pi(W(z_i)) \otimes \hat{R}(z_i)\right\| \tag{1.66}$$

$$= \sup_{\chi\in\hat{V}} \left\|\sum_i \lambda_i \pi(W(z_i))\chi(z_i)\right\| \tag{1.67}$$

$$(\omega \text{ non-deg.}) = \sup_{x\in V} \left\|\sum_i \lambda_i \pi(W(z_i))\mathrm{e}^{-\mathrm{i}\omega(x,z_i)}\right\| \tag{1.68}$$

$$(\mathrm{e}^{-\mathrm{i}\omega(x,z)}W(z) = W(x)W(z)W(x)^*) = \sup_{x\in V} \left\|\sum_i \lambda_i \pi(W(z_i))\right\| \tag{1.69}$$

$$= \sup_{x\in V} \left\|\sum_i \lambda_i \pi(W(z_i))\right\|\,. \tag{1.70}$$

□

Corollary 1.7.2. *There is a unique C^*-algebra $\tilde{\mathcal{W}}$ completing $\mathcal{W}$ up to isomorphism. It is simple. Every C^* seminorm on $\mathcal{W}$ is a norm, every representaion of $\tilde{\mathcal{W}}$ is isometric.*

1.8. Dynamics

Definition 1.8.1. A dynamics on $\mathcal{W}$ is a one-parameter group of automorphisms, $(\alpha_t \in \mathrm{Aut}(\mathcal{W}))_{t\in\mathbb{R}}$, i.e. $\alpha_0 = \mathrm{id}_{\mathcal{W}}$ and $\alpha_{t+s} = \alpha_t \circ \alpha_s$.

Proposition 1.8.2. *Let $\pi = (\pi, \mathcal{H})$ be an irreducible representation of $\mathcal{W}$. Suppose the dynamics is unitarily implemented, i.e. there exists a family $(U_t)_{t\in\mathbb{R}}$ of unitaries such that:*

$$\pi(\alpha_t(w)) = U(t)\pi(w)U(t)^*, \qquad w \in \mathcal{W}\,, \tag{1.71}$$

Suppose that $t \mapsto U(t)$ is continuous (in the sense of matrix elements) and differentiable, i.e. for some $\psi \in \mathcal{H}$ ∂_t exists in norm.

Then there exists a continuous one-parameter group of unitaries $t \mapsto V(t)$.

Remark 1.8.3. By Stone's theorem $V(t) = \mathrm{e}^{\mathrm{i}tH}$ and H plays the role of the Hamiltonian. While α_t is intrinsic, H depends on the representation.

Sometimes H can be used to distinguish representations. We will find that for a class of representations in low-dimensional conformal field theory, the knowledge of the discrete spectrum of $\mathcal{H}$ completely characterizes the representation category?!?.

Proof. We have

$$U(s)U(t)\pi(w)U(t)^*U(s)^* = U(s+t)\pi(w)U(s+t)^* \tag{1.72}$$

$$U(t)^*U(s)^*U(s+t)\pi(w) = \pi(w)U(t)^*U(s)^*U(s+t) \tag{1.73}$$

thus because π is irreducible, we get a function $\eta\colon \mathbb{R}\times\mathbb{R}\to\mathbb{T}$:

$$U(s+t) = \eta(s,t)U(s)U(t) \quad \leftrightarrow \quad \eta(s,t) = U(t)^*U(s)^*U(s+t) \in \mathrm{Hom}_{\mathcal{W}}(\pi,\pi) = \mathbb{C}\cdot 1\,. \tag{1.74}$$

We may assume that $U(0) = 1$, otherwise we multiply by a constant phase, thus we have

$$\eta(0,t) = \eta(s,0) \equiv 1 \tag{1.75}$$

Considering $U(r+s+t)$

$$U(r+t+s) = \eta(r,s+t)U(r)U(s+t) = \eta(r,s+t)\eta(s,t)U(r)U(s)U(t)\,, \tag{1.76}$$

$$U(r+s+t) = \eta(r+s,t)U(r+s)U(t) = \eta(r+s,t)\eta(r,s)U(r)U(s)U(t)\,. \tag{1.77}$$

thus η is a "coboundary":

$$\eta(r,s+t)\eta(s,t) = \eta(r+s,t)\eta(r,s) \tag{1.78}$$

which is symmetric $\eta(s,t) = \eta(t,s)$. If we show that it is actually a "coboundary"

$$\eta(s,t) = \frac{\xi(s)\xi(t)}{\xi(s+t)} \tag{1.79}$$

for some continuous function $\xi\colon \mathbb{R}\to\mathbb{T}$, then $V(t) = \xi(t)U(t)$ fulfills the property.

Let $\|\psi\| = 1$ such that $\partial_t U(t)\psi$ exists in norm, then

$$\eta(s,t) = U(t)^*U(s)^*U(s,t) = (U(t)\psi, U(s)^*U(s+t)\psi \tag{1.80}$$

shows that $\partial_t\eta(s,t)$ exists. Then

$$\xi(t) = \exp\left(\int_0^x \partial_2[\ln\eta(u,0)]\,\mathrm{d}u\right)$$

should do the job (Check!). □

Example 1.8.4 (The harmonich oscillator)**.** For the Heisenberg algebra we have heuristically

$$\alpha_t(Q) = \cos(\omega_0 t)Q - \sin(\omega_0 t)P\,, \tag{1.81}$$

$$\alpha_t(P) = \sin(\omega_0 t)Q + \cos(\omega_0 t)P\,. \tag{1.82}$$

In the Weyl setting this is given by

$$\alpha_t(W(z)) = W(\mathrm{e}^{\mathrm{i}t\omega_0 z}) \tag{1.83}$$

with $S_t z = \mathrm{e}^{\mathrm{i}t\omega_0}z$ and $c(z) = 0$. This dynamics is implemented in the Schrödinger representation by:

$$\pi_1(\alpha(w)) = U(t)\pi_1(w)U(t)^* \qquad U(t) = \mathrm{e}^{\mathrm{i}tH} \qquad H = \frac{P^2}{2m} + \frac{kQ^2}{2}, \omega_0 = \sqrt{\frac{k}{m}} \tag{1.84}$$

Example 1.8.5 (Free motion of a particle)**.** In the Heisenberg algebra we have:

$$\alpha_t(Q) = Q + \frac{t}{m}P\,, \tag{1.85}$$

$$\alpha_t(P) = P\,. \tag{1.86}$$

In the Weyl setting this is given by

$$\alpha_t(W(z)) = W(S_t z) \qquad S_t z = W(\mathrm{Re}\,z + (t/m + \mathrm{i})\,\mathrm{Im}\,z) \tag{1.87}$$

with S_t symplectic and $c(z) = 0$. This dynamics is implemented in the Schrödinger representation by:

$$\pi_1(\alpha(w)) = U(t)\pi_1(w)U(t)^* \qquad U(t) = \mathrm{e}^{\mathrm{i}tH} \qquad H = \frac{P^2}{2m} \tag{1.88}$$

Theorem 1.8.6 (No-go)**.** *Let* $H = \frac{P^2}{2m} + V(Q)$*,* $V \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$ *and* $U(t) = \mathrm{e}^{\mathrm{i}tH}$*, then*

$$U(t)\pi_1(w)U(t)^* \in \pi_1(\tilde{W}), W \in \tilde{W}, t \in \mathbb{R} \tag{1.89}$$

iff $V = 0$

—Lecture 3— —Lecture 4—

—Lecture 5— Goals:

- Understanding interesting unitary representations of the Poincaré group. (Prequantized theory)
- Understanding local aspects of such representations intrinisically, using ideas from Tomita-Takesaki's modular theory (modular theory)
- Associate quantum theory of free fields associated with this representations, basically using an infinite dimensional version of the Weyl/CCR algebra. (Second Quantization Functor)
- Abstracting what we learned to get an model independent approach (Haag-Kaaster Axioms)
- Discussing several physically interesting representations (so-called superselection sectors) for abstract quantum field theory (Doplicher–Haag–Roberts theory)

Definition 1.8.7. A unitary representation of $\widetilde{\mathcal{P}_+^\uparrow}$ is called **positive energy** if the joint spectrum of the generators of $U \restriction \mathbb{R}^4$, is contained in the forward light cone $\overline{V_+}$. This means $U(a) = \mathrm{e}^{\mathrm{i}aP} = \int_{\mathbb{R}^4} \mathrm{e}^{\mathrm{i}ap} \mathrm{d}E(p)$ and $\operatorname{supp} E \subset \overline{V_+}$.

Postulate [Wigner]: Irreducible positive energy representations correspond to particles.

Example 1.8.8 (Relativistic spin=0 particle with mass m)**.** Let $H_m := \{p \in \mathbb{R}^{1+3} : p^2 = m_2, p^0 > 0\}$. On $\mathcal{H}_{m,0} := L^2(H_m, d\mu)$ define the unitary representation:

$$[U(\Lambda, a)f)](p) = \mathrm{e}^{\mathrm{i}ap} f(\Lambda^{-1} p) \qquad (\Lambda, a) \in \widetilde{\mathcal{P}_+^\uparrow} \tag{1.90}$$

On $\mathbb{S}^1(\mathbb{R}^+1, \mathbb{R})$ define a representation of $\widetilde{\mathcal{P}_+^\uparrow}$ by $g_* f(x) = f(g^{-1}x) = f(\Lambda(A)^{-1}(x - a))$ for $g = (A, p) \in \widetilde{\mathcal{P}_+^\uparrow}$ and define $Ef = \hat{f} \restriction H_m$.

$$Eg_* f(p) = \frac{1}{(2\pi)^2} \int_{\mathbb{R}^4} \mathrm{e}^{\mathrm{i}xp} g_* f(x)\, \mathrm{d}x \tag{1.91}$$
$$= \mathrm{e}^{\mathrm{i}ap} Ef(\Lambda^{-1}(A)p) \tag{1.92}$$
$$= (U(g)Ef)(p) \tag{1.93}$$

so E intertwines the actions.

Joint spectrum: It is a tautology that U is a positive energy representation.
Kernel of E**:** Let $g = (\square_m^2)f$. Then

$$Eg = ((\mathrm{i}p_0)^2 - \sum_j (-\mathrm{i}p_j)^2 + m^2)f = 0 \tag{1.94}$$

Thus E factors through $\mathcal{S}(\mathbb{R}^{1+3}, \mathbb{R})/(\square + m^2)$, in particular we can identify $\mathcal{H}_{m,0}$ with solutions of the Klein-Gordon equation.

Causality: If $f, g\mathcal{S}(\mathbb{R}^{1+3}, \mathbb{R})$ with spacelike support, than $\operatorname{Im}(Ef, Eg) = 0$. (Exercise): The scalar product can be written as:

$$(f, g) = \mathrm{i} \int_{t=0} \left(\bar{g}\partial_0 f - \bar{f}' \partial_0 g\right) \mathrm{d}^3 x \tag{1.95}$$

where f, g are solutions of the Klein-Gordon equation.

1.9. Wigner's classification

Let U be an irreducible positive energy representation of $\widetilde{\mathcal{P}_+^\uparrow}$ on $\mathcal{H}$. From the commutation relations (semi-direct product) follows that the spectrum of P has to be Lorentz invariant.

$$U(A)^{-}1PU(A) = \Lambda(A)P$$

Since the representation is irreducible it should have only a single orbit under the action of $\widetilde{\mathcal{L}_+^\uparrow} = \mathsf{SL}(2,\mathbb{C})$.

1.9.1. **Massive case.** We fix $\bar{p} = (m,0,0,0)$ (particle in restframe). Let $F = F(\bar{p})$ the stabilizer of the point p. It is isomorphic to $\mathsf{SU}(2) \ltimes \mathbb{R}^4$.

Take the finite-dimensional representation $\mathsf{SU}(2) \rtimes \mathbb{R}^4$ on a finite-dimensional Hilbert space $\mathcal{K} = \mathrm{Sym}\otimes^n\mathbb{C}^2$, given by

$$\pi_F(U,a) = \mathrm{e}^{\mathrm{i}pa}D(A)$$

where $D(A) = A^{\otimes n}$, i.e. we choose an abitrary irreducible representation of the little group $\mathsf{SU}(2)$.

One can identify $\widetilde{\mathcal{P}_+^\uparrow}/F \cong H_m$ and one can choose a global section $s\colon H_m \to \widetilde{\mathcal{P}_+^\uparrow}$

$$s(p) = (H_p, 0) \qquad H_p = \frac{1}{\sqrt{2m(m+p_0)}}(m+\hat{p}) \tag{1.96}$$

Then:

$$s(p)^{-1}(A,a)s(\underbrace{\Lambda^{-1}(A)p}_{q}) = (H_p^{-1}AH_q, \Lambda(H_p)^{-1}a) \in F \tag{1.97}$$

On $L^2(H_m,\mathcal{K},\mu)$ with scalar product:

$$(\varphi_1,\varphi_2) = \int_{H_m}(\varphi_1(p),\varphi_2(p)_{\mathcal{K}}\mathrm{d}\mu(p) \tag{1.98}$$

define the unitary representation:

$$[U(g)\varphi](p) = \mathrm{e}^{\mathrm{i}pa}D(H_p^{-1}AH_q)\varphi(q)g = (A,p) \tag{1.99}$$

1.9.2. **Massless case.** $\bar{p} = (1,0,0,1)$ and $F = E(2) \ltimes \mathbb{R}_4$

$$E(2) = \left\{ L = \begin{pmatrix} \mathrm{e}^{\frac{\mathrm{i}}{2}\theta} & \mathrm{e}^{-\frac{\mathrm{i}}{2}\theta}z \\ 0 & \mathrm{e}^{\frac{\mathrm{i}}{2}\theta} \end{pmatrix} \in \mathsf{SL}(2,\mathbb{C}) : \theta \in [0,4\pi), z \in \mathbb{C} \right\}$$

Finite-dimensional representations of $E(2)$ are all 1-dimensional and non-faitful (finite helicity h). There are infinite dimensional representations (continuous spin rep).

$$\pi_F(L,a) = \mathrm{e}^{\mathrm{i}pa}\left(\mathrm{e}^{\frac{\mathrm{i}}{2}\theta}\right)^n$$

One can identify $\widetilde{\mathcal{P}_+^\uparrow}/F \cong \partial V_+$ and one can choose a Borel section $s\colon \partial V_+ \setminus p_3 = -p_0 \to \widetilde{\mathcal{P}_+^\uparrow}$

$$s(p) = (H_p, 0) \qquad H_p \begin{pmatrix} 2 & 0 \\ 0 & \\ 0 & \end{pmatrix} H_p^* = \hat{p} \tag{1.100}$$

On $L^2(\partial V_+,\mathcal{K},\mu)$ define the unitary representation:

$$[U_\pm(g)\varphi](p) = \mathrm{e}^{\mathrm{i}pa}\mathrm{e}^{\mathrm{i}pa}\left(\mathrm{e}^{\pm\frac{\mathrm{i}}{2}\theta}\right)^n\varphi(q)g = (A,p) \tag{1.101}$$

with helicity $h = \pm n/2$.