LECTURE NOTES: ALGEBRAIC AND TOPOLOGICAL QUANTUM FIELD THEORY CHAPTER 3: FREE NETS ON THE CIRCLE

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Abstract. ATTENTION: not proof read lecture notes in progress.....

Contents

1.1. Second quantization

We want to introduce the Bosonic and Fermionic second quantization. Given a Hilbert space $\mathcal{H}$, the one-particle Hilbert space, and a real subspace $H \subset \mathcal{H}$ one can associate two von Neumann algebras: the Bosonic $R(H)$ and the Fermionic $C(H)$ which are acting on the symmetric (Bosonic) and antisymmetric (Fermionic) Fock space, respectively. Unitaries on the one-parameter space $\mathcal{H}$ can be promoted to second quantization unitaries on the respective Fock spaces and it turns out that if H is standard then Fock space vacuum vector Ω is cyclic and separating, in other words $(R(H), \Omega)$ and $(C(H), \Omega)$ happen to be in standard form and the modular unitary groups are given by second quantization of the modular unitary group of the standard subspace H. Further it holds some abstract version Haag duality in the Bose case or Haag–Araki duality in the Fermi case, namely $R(H') = R(H)'$ and $C(H^\perp) = C(H)^\sharp$ where $H^\perp = \mathrm{i}H'$ is the real orthogonal complement and $M^\sharp$ is the graded commutant. Some generalization will be given in Section **??**.

Date: today.

1.1.1. **Bosonic.** Let $\mathcal{H}$ be a Hilbert space and $\omega(\,\cdot\,,\,\cdot\,) = \mathrm{Im}(\,\cdot\,,\,\cdot\,)$ the sesquilinear form. Then we can define unitaries $W(f)$ for $f \in \mathcal{H}$ fulfilling

$$W(f)W(g) = \mathrm{e}^{-\mathrm{i}\omega(f,g)}W(f+g) = \mathrm{e}^{-2\mathrm{i}\omega(f,g)}W(g)W(f)$$

and acting naturally on the Bosonic Fock space $\mathrm{e}^{\mathcal{H}}$ over $\mathcal{H}$. This space is given by $\mathrm{e}^{\mathcal{H}} = \sum_{n=0}^{\infty} P_n\mathcal{H}^{\otimes n}$, where P_n is the projection $P_n(x_1 \otimes \cdots \otimes x_n) = 1/n! \sum_\sigma x_{\sigma(1)} \otimes \cdots \otimes x_{\sigma(n)}$ and the sum goes through all permutations. The set of coherent vectors $\mathrm{e}^h := \bigoplus_{n=0}^{\infty} h^{\otimes n}/\sqrt{n!}$ with $h \in \mathcal{H}$ is total in $\mathrm{e}^{\mathcal{H}}$ and it is $(\mathrm{e}^f, \mathrm{e}^h) = \mathrm{e}^{(f,h)}$. The vacuum is given by $\Omega = \mathrm{e}^0$ and the action of $W(f)$ is given by $W(f)\mathrm{e}^0 = \mathrm{e}^{-\frac{1}{2}\|f\|^2}\mathrm{e}^f$, in other words the vacuum representation $\omega(\,\cdot\,) = (\Omega,\,\cdot\,\Omega)$ is characterized by $\omega(W(f)) = \mathrm{e}^{-\frac{1}{2}\|f\|^2}$.

Finally, for a real subspace $H \subset \mathcal{H}$ we define the von Neumann algebra

$$R(H) = \{W(f) : f \in H\}'' \subset \mathrm{B}(\mathrm{e}^{\mathcal{H}})\,.$$

Let U be a unitary in $\mathrm{B}(\mathcal{H})$ then $\Gamma(U) = \bigoplus_{n=0}^{\infty} U^{\otimes n}$ acts on coherent states by $\Gamma(U)\mathrm{e}^h = \mathrm{e}^{Uh}$ and is therefore a unitary (cf. [Gui11]) on $\mathrm{e}^{\mathcal{H}}$, the **second quantization unitary** of U. Second quantization unitaries implement Boguliubov automorphism, namely $\Gamma(U)W(f)\Gamma(U)^* = W(Uf)$ and in particular we have covariance $\Gamma(U)R(H)\Gamma(U)^* = R(UH)$. The map R has the following properties:

Proposition 1.1.1 ([Lon08])**.**

(1) Let $H, K \subset \mathcal{H}$ be real linear subspaces. Then $R(K) = R(H)$ iff $\bar{K} = \bar{H}$.
(2) Let H be closed. H is separating or cyclic iff $R(H)$ is separating or cyclic, respectively.
(3) Let H be standard, then the modular unitaries $\Delta^{\mathrm{i}t}_{(R(H),\Omega)}$ and the modular conjugation $J_{(R(H),\Omega)}$ associated with $(R(H), \Omega)$ are given by

$$\Delta^{\mathrm{i}t}_{(R(H),\Omega)} = \Gamma(\Delta^{\mathrm{i}t}_H), \qquad J_{(R(H),\Omega)} = \Gamma(J_H)$$

and in particular $R(H') = R(H)'$.

1.1.2. **Fermionic.** Let $\mathcal{H}^1$ be a complex Hilbert space and $\mathcal{H} = \Lambda(\mathcal{H}^1)$ the antisymmetric or Fermionic Fock space is obtained by completing the exterior algebra with the inner product:

$$(e_1 \wedge \cdots \wedge e_m, f_1 \wedge \cdots \wedge f_n) = \delta_{mn} \det A^{\mathbf{e},\mathbf{f}} \qquad \text{where } A^{\mathbf{e},\mathbf{f}}_{ij} = (e_i, f_j) \text{ for } 1 \leq i,j \leq n\,.$$

For $A \in \mathrm{B}(\mathcal{H}^1)$ with $\|A\| \leq 1$ we define $\Lambda(A)$ to be $A^{\otimes k}$ on $\mathcal{H}^k := \Lambda^k(\mathcal{H}^1) \subset (\mathcal{H}^1)^{\otimes k}$. If U is a unitary we call $\Lambda(U)$ **second quantization unitary**. The space is $\mathbb{Z}_2$ graded by $\Gamma := \Lambda(-1)$. We define $Z = \frac{1-\mathrm{i}\Gamma}{1-\mathrm{i}}$ and note that $Z^2 = \Gamma$. For $f \in \mathcal{H}^1$ let $a(f)$ be the bounded operator obtained by continuing the exterior multiplication $f \wedge \cdot$. The operators fulfill the complex Clifford relations $a(f)^*a(g) + a(g)a(f)^* = (f,g)$ and $\{a(f), a(g)\} = \{a(f)^*, a(g)^*\} = 0$ for all $f, g \in \mathcal{H}^1$. For a standard subspace $K \subset \mathcal{H}^1$ we define the von Neumann algebra

$$C(K) = \{c(f) : f \in K\}'' \subset \mathrm{B}(\mathcal{H})$$

where $c(f) = a(f) + a(f)^*$, which fulfills the real Clifford relations $c(f)c(g) + c(g)c(f) = 2\,\mathrm{Re}(f,g)$. By $\Omega = 1 \in \Lambda^0(\mathcal{H}^1)$ we denote the vacuum which is cyclic and separating for $C(K)$ for every standard subspace $K \subset \mathcal{H}^1$. For a real subspace K we define the **real orthogonal complement** to be $K^\perp = \mathrm{i}K' = \{x \in \mathcal{H} : \mathrm{Re}(x, K) = 0\}$.

Proposition 1.1.2 ([Foi83, Was98])**.**

(1) Let $H, K \subset \mathcal{H}^1$ be real linear subspaces. Then $C(K) = C(H)$ iff $\bar{K} = \bar{H}$.
(2) Let H be closed. H is separating or cyclic iff $C(H)$ is separating or cyclic, respectively.

(3) *Let H be standard, then the modular unitaries $\Delta^{\mathrm{i}t}_{(C(H),\Omega)}$ and the modular conjugation $J_{(C(H),\Omega)}$ associated with $(R(H),\Omega)$ are given by*

$$\Delta^{\mathrm{i}t}_{(C(H),\Omega)} = \Lambda(\Delta^{\mathrm{i}t}_H), \qquad J_{(C(H),\Omega)} = \hat{\Lambda}(J_H) := Z^*\Lambda(\mathrm{i}J_H)$$

*and in particular it holds **Haag-Araki duality**, i.e. $C(K^\perp)$ equals $C(K)^\perp := ZC(K)'Z^*$, the **twisted commutant** of $C(K)$.*

For a unitary U on $\mathcal{H}^1$ it holds $\Lambda(U)c(f)\Lambda(U^*) = c(Uf)$, which implies that C is covariant with respect to the unitaries $U(\mathcal{H}^1)$, i.e. $\Lambda(U)C(K)\Lambda(U)^* = C(UK)$.

We note that for example in the case of the complex Fermion the one-particle space is obtained from a Hilbert space $\mathcal{H}^1$ (the space of test functions) and a projection P by $\mathcal{H}^1_P = P\mathcal{H}^1 \oplus \overline{P^\perp\mathcal{H}^1}$ and one gets a new representation of the complex Clifford algebra on $\Lambda(\mathcal{H}^1_P)$ by defining

$$a_P(f) := a(Pf) + a(\overline{P^\perp f})^*$$

where $a(f)$ is the creation operator. For a standard subspace $K \subset \mathcal{H}^1_P$ which is invariant under the multiplication of $\mathrm{i}_{\mathcal{H}^1}$ in $\mathcal{H}^1$, the von Neumann algebra $C(K)$ on $\Lambda(\mathcal{H}^1_P)$ coincides with the von Neumann algebra $\{a_P(f), a_P(f)^* : f \in K\}''$, so in this case

$$R(H) = \{c(f) : f \in H\}'' = \{a_P(f), a_P(f)^* : f \in K\}''$$

holds. Indeed, the one inclusion follows from $c(f) = a_P(f) + a_P(f)^*$ and the other follows from Araki-Haag duality and $\{a_P(f), c(g)\} = (g,f)_{\mathcal{H}^1} = \mathrm{Re}(g,f)_{\mathcal{H}^1_P} - \mathrm{i}\,\mathrm{Re}(g, \mathrm{i}_{\mathcal{H}^1} f)_{\mathcal{H}^1_P} = 0$ for $f \in K$ and $g \in K^\perp$. We further note that the space $\Lambda(\mathcal{H}^1_P)$ is as a real Hilbert space the same as $\Lambda(\mathcal{H}^1)$ and can be identified canonically with $\Lambda(P\mathcal{H}^1) \otimes \Lambda(\overline{P^\perp\mathcal{H}^1})$.

1.2. Boson–Fermion Correspondence

We have introduced the Bosonic and Fermionic second quantization. We want to see that for nets on the circle these are related.

1.2.1. **Möbius group.**

The chiral parts of 2D chiral conformal (or Möbius covariant) quantum field theory can be seen as two quantum field theories on the light rays. They extends to a theory on the compactification of one light ray, which is the circle. This motivates to look into quantum field theory on the circle. The symmetry group which leaves the vacuum state invariant turns out to be the Möbius group. We give some basics about the Möbius group and its positive energy representation (discrete series).

We are concerned with the **circle** $\mathbb{S}^1$ as a spacetime and the Möbius group $\mathsf{M\ddot{o}b}$ as symmetry group of $\mathbb{S}^1$. The compactified real line $\overline{\mathbb{R}}$ can be identified with the circle $\mathbb{S}^1 = \{z \in \mathbb{C} : |z| = 1\}$ by the Caley map $C : \overline{\mathbb{R}} \longrightarrow \mathbb{S}^1$

$$C(x) = -\frac{x-\mathrm{i}}{x+\mathrm{i}} = \mathrm{e}^{\mathrm{i}2\arctan x} \quad \Longleftrightarrow \quad x = C^{-1}(z) = -\mathrm{i}\frac{z-1}{z+1},\ x = C^{-1}(\mathrm{e}^{\mathrm{i}\theta}) = \tan\frac{\theta}{2}\,.$$

We speak about the **circle picture** and **real line picture**, respectively. The **Möbius group** $\mathsf{M\ddot{o}b}$ can naturally be identified with $\mathsf{PSU}(1,1)$ in the circle picture:

$$\mathsf{PSU}(1,1) = \mathsf{SU}(1,1)/\{+I,-I\}\,, \qquad \mathsf{SU}(1,1) = \left\{\begin{pmatrix}\alpha & \beta\\ \bar\beta & \bar\alpha\end{pmatrix} \in \mathsf{GL}(2,\mathbb{C}) : |\alpha|^2 - |\beta|^2 = 1\right\}$$

by the action

$$g : \mathbb{S}^1 \longrightarrow \mathbb{S}^1\,, \qquad z \longmapsto gz = \frac{\alpha z + \beta}{\bar\beta z + \bar\alpha}\,.$$

In the real line picture $\mathsf{M\ddot{o}b}$ can be identified with $\mathsf{M\ddot{o}b} \cong \mathsf{PSL}(2,\mathbb{R})$

$$\mathsf{PSL}(2,\mathbb{R}) = \mathsf{SL}(2,\mathbb{R})/\{+I,-I\}\,, \qquad \mathsf{SL}(2,\mathbb{R}) = \left\{\begin{pmatrix} a & b \\ c & d\end{pmatrix} \in \mathsf{GL}(2,\mathbb{R}) : \det A = ad - bc = 1\right\}$$

which acts naturally on the compactified real line $\overline{\mathbb{R}}$ with the action given by

$$g : \overline{\mathbb{R}} \longrightarrow \overline{\mathbb{R}}\,, \qquad x \longmapsto gx = \frac{ax+b}{cx+d}\,.$$

The following subgroups of $\mathsf{M\ddot{o}b}$: **rotations** $\{R(\vartheta)\}_{\vartheta\in\mathbb{R}/2\pi\mathbb{Z}}$, **dilations** $\{\delta(s)\}_{s\in\mathbb{R}}$ and **translations** $\{\tau(t)\}_{t\in\mathbb{R}}$ are given by

$$R(\vartheta) = \begin{pmatrix} \mathrm{e}^{\mathrm{i}\frac{\vartheta}{2}} & 0 \\ 0 & \mathrm{e}^{-\mathrm{i}\frac{\vartheta}{2}}\end{pmatrix} \qquad \delta(s) = \begin{pmatrix} \cosh\frac{s}{2} & -\sinh\frac{s}{2} \\ -\sinh\frac{s}{2} & \cosh\frac{s}{2}\end{pmatrix} \qquad \tau(t) = \begin{pmatrix} 1-\frac{t}{2\mathrm{i}} & -\frac{t}{2\mathrm{i}} \\ \frac{t}{2\mathrm{i}} & 1+\frac{t}{2\mathrm{i}}\end{pmatrix}$$

as subgroups of $\mathsf{PSU}(1,1)$ and by

$$R(\vartheta) = \begin{pmatrix} \cos\frac{\vartheta}{2} & \sin\frac{\vartheta}{2} \\ -\sin\frac{\vartheta}{2} & \cos\frac{\vartheta}{2}\end{pmatrix} \qquad \delta(s) = \begin{pmatrix} \mathrm{e}^{\frac{s}{2}} & 0 \\ 0 & \mathrm{e}^{-\frac{s}{2}}\end{pmatrix} \qquad \tau(t) = \begin{pmatrix} 1 & t \\ 0 & 1\end{pmatrix}$$

as subgroups of $\mathsf{PSL}(2,\mathbb{R})$. The action of these subgroups in the circle and real line pictures are given by

$$\begin{aligned} R(\vartheta)z &= \mathrm{e}^{\mathrm{i}\vartheta}z\,, & R(\vartheta)x &= \tan\left(\tfrac{\vartheta}{2} + \arctan(x)\right)\,, \\ \delta(s)z &= \frac{z\cosh\frac{s}{2} - \sinh\frac{s}{2}}{\cosh\frac{s}{2} - z\sinh\frac{s}{2}}\,, & \delta(s)x &= e^s x\,, \\ \tau(t)z &= \frac{z - \frac{t}{2\mathrm{i}}(z+1)}{1+\frac{t}{2\mathrm{i}}(z+1)}\,, & \tau(t)x &= x+t\,, \end{aligned}$$

respectively, where $z \in \mathbb{S}^1$ and $x \in \overline{\mathbb{R}}$. For rotations the circle picture is easier, while for dilations and translations the real line picture is easier.

1.2.2. **Positive energy representations of** $\mathsf{M\ddot{o}b}$**.** Let U be a unitary representation of $\mathsf{M\ddot{o}b}$ on $\mathcal{H}$. It is called **positive energy representation** of $\mathsf{M\ddot{o}b}$ if the generator L_0 of the rotations $U(R(\vartheta)) = \mathrm{e}^{\mathrm{i}\vartheta L_0}$ has positive spectrum. The rotation subgroup $U(R(\vartheta))$ defines a grading on $\mathcal{H}$

$$\mathcal{H} = \bigoplus_{n=0}^{\infty} \mathcal{H}_n\,, \qquad \mathcal{H}_n = \{x \in \mathcal{H} : U(R(\vartheta))x = \mathrm{e}^{\mathrm{i}n\vartheta}x\}$$

and there exists a number m such that $\mathcal{H}_m \neq \{0\}$ and $\mathcal{H}_n = \{0\}$ for $n < m$ which is called the **lowest weight** of U.

Theorem 1.2.1. *For each $m \in \mathbb{N}$ there exists a unique irreducible representation of* $\mathsf{M\ddot{o}b}$ *with lowest weight m. Every positive energy of* $\mathsf{M\ddot{o}b}$ *representation is a direct sum of irreducible representations.*

Example 1.2.2 (Lowest weight 1)**.** Consider $C^\infty(\mathbb{S}^1,\mathbb{R})$, where we write the periodic function $f \in C^\infty(\mathbb{S}^1,\mathbb{R})$ as a Fourier series

$$f(\theta) = \sum_{k\in\mathbb{Z}} \hat{f}_k \mathrm{e}^{\mathrm{i}k\theta}, \qquad \hat{f}_k = \int_0^{2\pi} \mathrm{e}^{-\mathrm{i}k\theta} f(\theta)\frac{\mathrm{d}\theta}{2\pi} = \overline{\hat{f}_{-k}}\,.$$

We introduce a semi-norm

$$\|f\|^2 = \sum_{k=1}^{\infty} k\cdot|\hat{f}_k|^2$$

and a complex structure, i.e. an isometry $\mathcal{J}$ with respect to $\|\cdot\|$ satisfying $\mathcal{J}^2 = -1$, by $\mathcal{J} : \hat{f}_k \mapsto -\mathrm{i}\,\mathrm{sign}(k)\hat{f}_k$. Finally, we get the Hilbert space

$$\mathcal{H} = \overline{C^\infty(\mathbb{S}^1,\mathbb{R})/\mathbb{R}}^{\|\cdot\|}$$

by completion with respect to the norm $\|\cdot\|$, where $\mathbb{R}$ is identified with the constant functions. By abuse of notation we denote also the image of $f \in C^\infty(\mathbb{S}^1,\mathbb{R})$ in $\mathcal{H}$ by f. The scalar product (linear in the second component) and the sesquilinear form $\omega(\,\cdot\,,\,\cdot\,) \equiv \mathrm{Im}(\,\cdot\,,\,\cdot\,)$ are given by

$$(f,g) = \sum_{k=1}^{\infty} k\hat{f}_k\hat{g}_{-k}\,, \qquad \omega(f,g) = \frac{-\mathrm{i}}{2}\sum_{k\in\mathbb{Z}} k\hat{f}_k\hat{g}_{-k} = \frac{1}{2}\int_0^{2\pi} f(\theta)g'(\theta)\frac{\mathrm{d}\theta}{2\pi} = \frac{1}{4\pi}\int f\mathrm{d}g\,,$$

respectively. The unitary action U_1 of $\mathsf{M\ddot{o}b}$ on $\mathcal{H}$ is induced by the action on $C^\infty(\mathbb{S}^1,\mathbb{R})$ given by

$$(U_1(g)f)(z) := f(g^{-1}(z))\,.$$

1.2.3. **Double cover and universal cover of** $\mathsf{M\ddot{o}b}$**.** We denote by $\mathsf{M\ddot{o}b}^{(2)} \cong \mathsf{SU}(1,1) \cong \mathsf{SL}(2,\mathbb{R})$ the double cover of $\mathsf{M\ddot{o}b}$:

$$1 \to \mathbb{Z}_2 \to \mathsf{M\ddot{o}b}^{(2)} \to \mathsf{M\ddot{o}b} \to 1$$

which acts faithful on the double cover $\mathbb{S}^{1(2)} \cong \mathbb{R}/4\pi\mathbb{Z}$ of the circle, with the projection $\mathbb{R}/4\pi\mathbb{Z} \to \mathbb{S}^1 : \theta \mapsto \mathrm{e}^{\mathrm{i}\theta}$. The action of the rotation is given by $R(\vartheta)\theta = \theta + \vartheta$.

We denote by $\widetilde{\mathsf{M\ddot{o}b}}$ the universal covering of $\mathsf{M\ddot{o}b}$:

$$1 \longrightarrow \mathbb{Z} \longrightarrow \widetilde{\mathsf{M\ddot{o}b}} \longrightarrow \mathsf{M\ddot{o}b} \longrightarrow 1$$

which acts on the universal cover $\mathbb{R}$ of the circle with $\pi : \mathbb{R} \to \mathbb{S}^1 : \theta \mapsto \mathrm{e}^{\mathrm{i}\theta}$ and the rotation act on $\mathbb{R}$ also by $\mathrm{R}(\vartheta)\theta = \theta + \vartheta$.

Let $j : z \longmapsto \bar{z}$ the reflection on the circle. It acts by $\rho_j(g) = jgj$ on and we can define $\mathsf{M\ddot{o}b}_2 = \mathsf{PSL}(2,\mathbb{R})_\pm = \mathsf{PSL} \rtimes_{\rho_j} \mathbb{Z}_2$ which turns out to be isomorphic to

$$\begin{aligned} \mathsf{PSL}(2,\mathbb{R})_\pm &\cong \mathsf{SL}(2,\mathbb{R})_\pm/\{+I,-I\}\,,\\ \mathsf{SL}(2,\mathbb{R})_\pm &= \left\{\begin{pmatrix} a & b\\ c & d\end{pmatrix} \in \mathsf{GL}(2,\mathbb{R}) : \det A = ad-bc = \pm 1\right\} \end{aligned}$$

and has two connected components $\mathsf{PSL}(2,\mathbb{R})_+ = \mathsf{PSL}(2,\mathbb{R})$ and its coset $\mathsf{PSL}(2,\mathbb{R})_-$ given by ρ_j. ρ_j lifts to $\mathsf{M\ddot{o}b}^{(2)}$ and $\widetilde{\mathsf{M\ddot{o}b}}$ and we can define analogously $\mathsf{M\ddot{o}b}_2^{(2)} \cong \mathsf{SL}(2,\mathbb{R})_\pm$ and $\widetilde{\mathsf{M\ddot{o}b}}_2 = \widetilde{\mathsf{M\ddot{o}b}} \rtimes \mathbb{Z}_2$. Following groups are involved:

$$\begin{array}{ccccccccc}
 & & 1 & & 1 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & \mathbb{Z} & = & \mathbb{Z} & & & & \\
 & & \downarrow & & \downarrow & & & & \\
1 & \longrightarrow & \widetilde{\mathsf{M\ddot{o}b}} & \longrightarrow & \widetilde{\mathsf{M\ddot{o}b}}_2 & \longrightarrow & \mathbb{Z}_2 & \longrightarrow & 1 \\
 & & \downarrow{\scriptstyle \pi_{\widetilde{\mathsf{M\ddot{o}b}}}} & & \downarrow{\scriptstyle \pi_{\widetilde{\mathsf{M\ddot{o}b}}_2}} & & \| & & \\
1 & \longrightarrow & \mathsf{M\ddot{o}b} & \longrightarrow & \mathsf{M\ddot{o}b}_2 & \longrightarrow & \mathbb{Z}_2 & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 1 & & 1 & & & &
\end{array}\quad.$$

and we get a similar picture with $\widetilde{\mathsf{M\ddot{o}b}}, \mathsf{M\ddot{o}b}_2, \mathbb{Z}$ replaced by $\mathsf{M\ddot{o}b}^{(2)}, \mathsf{M\ddot{o}b}^{(2)}, \mathbb{Z}_2$. We denote again the subgroups of rotations, dilations and translations by $R(\vartheta)$, $\delta(t)$ and $\tau(s)$, respectively. We remark that $\widetilde{\mathsf{M\ddot{o}b}}$ has no representation as matrix group.

1.3. Fermi Nets

Let $\mathcal{I}$ be the set of proper intervals $I \subset \mathbb{S}^1$, i.e. open, connected, non-empty, non-dense intervals.

Definition 1.3.1. A **graded local Möbius covariant net** or **Fermi net** $\mathcal{F}$ on $\mathbb{S}^1$ is a family $\{\mathcal{F}(I)\}_{I\in\mathcal{I}}$ of von Neumann algebras on a $\mathbb{Z}_2$-graded Hilbert space $\mathcal{H}$, i.e. there is a unitary operator Γ with $\Gamma^2 = 1$ with the following properties:

(1) **Isotony.** $I_1 \subset I_2$ implies $\mathcal{F}(I_1) \subset \mathcal{F}(I_2)$.
(2) **Graded locality.** $I_1 \cap I_2 = \emptyset$ implies $[\mathcal{F}(I_1), Z\mathcal{F}(I_2)Z^*] = \{0\}$, where $Z = \frac{1-\mathrm{i}\Gamma}{1-\mathrm{i}}$.
(3) **Möbius covariance.** There is a unitary representation U of $\mathsf{M\ddot{o}b}^{(2)}$ on $\mathcal{H}$ such that $U(g)\mathcal{F}(I)U(g)^* = \mathcal{F}(gI)$.
(4) **Positivity of energy.** U is a positive energy representation, i.e. the generator L_0 (conformal Hamiltonian) of the rotation subgroup $U(R(\theta)) = \mathrm{e}^{\mathrm{i}\theta L_0}$ has positive spectrum.
(5) **Vacuum.** There is a (up to phase) unique rotation invariant and even (i.e. $\Gamma\Omega = \Omega$) unit vector $\Omega \in \mathcal{H}$ which is cyclic for the von Neumann algebra $\bigvee_{I\in\mathcal{I}} \mathcal{F}(I)$.

If $Z = 1$ then the net is called a **local Möbius covariant net** or **conformal net** Among the consequences of the definition are ([CKL08]):

Proposition 1.3.2. *Let $\mathcal{F}$ be a Fermi net on $\mathcal{H}$. Then the following holds.*

(1) ***Reeh–Schlieder property.*** *For each $I \in \mathcal{I}$ the vector Ω is cyclic and separating for each $\mathcal{F}(I)$.*
(2) ***Additivity.*** *If $I = \bigcup_i I_i$, then $\mathcal{F}(I) = \bigvee_i \mathcal{F}(I_i)$.*
(3) ***Bisognano–Wichmann property.*** *The modular group $\Delta^{\mathrm{i}t}_{(\mathcal{F}(I),\Omega)}$ is given by $U(\delta_I(-2\pi t))$.*
(4) ***Twisted Haag duality*** *or* ***Haag–Araki duality.*** *For $I \in \mathcal{I}$ we have $\mathcal{F}(I)' = Z\mathcal{F}(I')Z^*$.*
(5) ***Irreducibility.*** *We have $\bigvee_{I\in\mathcal{I}} \mathcal{F}(I) = \mathrm{B}(\mathcal{H})$.*

1.4. Examples

1.5. Boson–Fermion correspondence

Here we will use the notion of a Fermi net (Section 1.3). Let $\mathcal{F}$ be a Fermi net. The representation U of $\mathsf{M\ddot{o}b}^{(2)}$ restricts to a projective unitary representation of $\mathsf{M\ddot{o}b}$ [CKL08]. We denote by $R(\theta) = U(R(\theta))$ the rotation subgroup, where $R(\theta)$ is the 4π-periodic lift of the rotation from $\mathsf{M\ddot{o}b}$ to $\mathsf{M\ddot{o}b}^{(2)}$. Further we denote the subgroup of translations by $T(t) = U(\tau(t))$, where τ is the lift of the translation from $\mathsf{M\ddot{o}b}$ to $\mathsf{M\ddot{o}b}^{(2)}$.

1.5.1. **Subnets and the character argument.** Let $\mathcal{F}$ be a Fermi (or conformal) net on $\mathcal{H}_\mathcal{F}$. Another assignment $\mathcal{A}$ of von Neumann algebras $\{\mathcal{A}(I)\}_{I\in\mathcal{I}}$ on $\mathcal{H}_\mathcal{F}$ is called like in the local case (cf. Subsection **??**) a **subnet** of $\mathcal{F}$ if it satisfies isotony, Möbius covariance with respect to the same U for $\mathcal{F}$ and it holds that $\mathcal{A}(I) \subset \mathcal{F}(I)$ for every interval $I \in \mathcal{I}$. We simply write $\mathcal{A} \subset \mathcal{F}$. In this case, let us denote $\mathcal{H}_\mathcal{A} = \overline{\bigvee_{I\in\mathcal{I}} \mathcal{A}(I)\Omega}$.

Then it is immediate to see that $\mathcal{A}(I)$ and U restrict to $\mathcal{H}_\mathcal{A}$, and by this restriction $\mathcal{A}\restriction_{\mathcal{H}_\mathcal{A}}$ becomes a Fermi net.

For a Fermi net $\mathcal{F}$ on $\mathcal{S}$, a **gauge automorphism** α is a family of automorphisms $\{\alpha_I\}$ of local algebras which satisfies the consistency condition

$$\alpha_{I_2}|_{\mathcal{A}(I_1)} = \alpha_{I_1} \qquad\qquad \text{for } I_1 \subset I_2\,.$$

If a gauge automorphism α preserves the vacuum state $(\Omega,\ \cdot\ \Omega)$, it is said to be an **inner symmetry**. An inner symmetry α can be unitarily implemented by the formula $V_\alpha x\Omega = \alpha(x)\Omega$, where x is an element of some local algebra $\mathcal{F}(I)$. We say that a *compact* group G acts on the net $\mathcal{F}$ if there are automorphisms $\{\alpha_g\}_{g\in G}$ which satisfy the composition law when restricted to local algebras. The **fixed point subnet** with respect to this action of G is the subnet defined by $\mathcal{F}^G(I) := \mathcal{F}(I)^G$.

Let $\mathcal{F}$ be a Fermi net and $\mathcal{A}$ be a subnet. Recall that, for a Möbius covariant Fermi net, the Bisognano-Wichmann property is automatic. As a consequence, for each interval there is a conditional expectation $E_I : \mathcal{F}(I) \to \mathcal{A}(I)$ which preserves the vacuum state $(\Omega,\ \cdot\ \Omega)$ and is implemented by the projection $P_\mathcal{A}$ onto $\mathcal{H}_\mathcal{A}$ (by Theorem **??**).

Consider the case where $\mathcal{A} = \mathcal{F}^G$ is the fixed point subnet with respect to an action α of a compact group G by inner symmetry. Then we have a unitary representation V_α of G on $\mathcal{H}_\mathcal{F}$. If we write the set of invariant vectors with respect to V_α by $\mathcal{H}_\mathcal{F}^G$, it holds that $\mathcal{H}_\mathcal{F}^G = \mathcal{H}_\mathcal{A}$. Indeed, the inclusion $\mathcal{H}_\mathcal{A} \subset \mathcal{H}_\mathcal{F}^G$ is obvious. On the other hand, for $x \in \mathcal{F}(I)$, we have

$$\left(\int_G \alpha(x)\mathrm{d}g\right)\Omega = \int_G (V_\alpha(g)x\Omega)\,\mathrm{d}g,$$

which implies that any vector in $\mathcal{H}_\mathcal{F}^G$ can be approximated from $\mathcal{H}_\mathcal{A}$ by the Reeh–Schlieder property.

For the later use, we put here a simple observation.

Proposition 1.5.1. *In the situation above, if a Longo–Witten endomorphism is implemented by W and W commutes with V_α, then* Ad W *restricts to a Longo–Witten endomorphism of the fixed point subnet $\mathcal{A}$.*

Proof. The unitary W commutes with the projection $P_{\mathcal{A}_0}$, hence also with the conditional expectation E onto $\mathcal{A}$, cf. Lemma **??**. □

Let $\mathcal{F}$ be Fermi (or local) net on $\mathcal{H}_\mathcal{F}$. The Hilbert space $\mathcal{H}_\mathcal{F}$ is graded by the action of the rotation subgroup $R(\theta) = \mathrm{e}^{\mathrm{i}\theta L_0}$:

$$\mathcal{H}_\mathcal{F} = \mathbb{C}\Omega \oplus \bigoplus_{r\in\frac{1}{2}\mathbb{N}} \mathcal{H}_r = \bigoplus_{r\in\frac{1}{2}\mathbb{N}_0} \mathcal{H}_r$$

with $\mathcal{H}_r = \{\xi \in \mathcal{H}_\mathcal{F} : R(\theta)\xi = \mathrm{e}^{\mathrm{i}r\theta}\xi\}$ and the sum only going over $\mathbb{N}_0$ for a local net. The **conformal character** of the net $\mathcal{F}$ is given as a formal power series of $t = \mathrm{e}^{-\beta}$:

$$\operatorname{tr}_{\mathcal{H}_\mathcal{F}}(\mathrm{e}^{-\beta L_0}) = \sum_{r\in\frac{1}{2}\mathbb{N}_0}^{\infty} \dim \mathcal{H}_r \cdot t^r \,.$$

Let us assume that there is an action of $G = \mathsf{U}(1)$ by inner symmetry. We denote by $V(\theta)$ the implementing unitary. Then V and U commute and $\mathcal{H}_\mathcal{F}$ is graded also by the gauge action $V(\theta) = \mathrm{e}^{\mathrm{i}\theta Q}$:

$$\mathcal{H}_\mathcal{F} = \mathbb{C}\Omega_0 \oplus \bigoplus_{r\in\frac{1}{2}\mathbb{N},q\in\mathbb{Z}} \mathcal{H}_{r,q} = \bigoplus_{q\in\mathbb{Z}} \mathcal{H}_{\,\cdot\,,q}, \qquad \text{with} \qquad \mathcal{H}_{\,\cdot\,,q} := \bigoplus_{r\in\frac{1}{2}\mathbb{N}_0} \mathcal{H}_{r,q}$$

and the character is given as a formal power series in $t = \mathrm{e}^{-\beta}$ and $z = \mathrm{e}^{-E}$:

$$\operatorname{tr}_{\mathcal{H}_\mathcal{F}}(\mathrm{e}^{-\beta L_0 - EQ}) = \sum_{r\in\frac{1}{2}\mathbb{N}_0,q\in\mathbb{Z}} \dim \mathcal{H}_{r,q} \cdot t^r z^q \,.$$

Recall that it holds that $\mathcal{H}_\mathcal{F}^G = \mathcal{H}_\mathcal{A}$. The operator Q acts by 0 on $\mathcal{H}_\mathcal{F}^G$, hence we can obtain the conformal character of $\mathcal{A}$ just by taking the coefficient of z^0 in $\operatorname{tr}_{\mathcal{H}_\mathcal{F}}(\mathrm{e}^{-\beta L_0 - EQ})$.

Later in this section we need to compare the size of two subnets. Let $\mathcal{A} \subset \mathcal{B} \subset \mathcal{F}$ be an inclusion of three Fermi nets. If the conformal characters of $\mathcal{A}$ and $\mathcal{B}$ coincide, then this means that the subspaces $\mathcal{H}_\mathcal{A}$ and $\mathcal{H}_\mathcal{B}$ coincide, since we have already an inclusion $\mathcal{H}_\mathcal{A} \subset \mathcal{H}_\mathcal{B}$ and the coefficients of the conformal character are the dimensions of eigenspaces of L_0. This in turn implies that two subnets $\mathcal{A}$ and $\mathcal{B}$ are the same by the given argument in the proof of Lemma **??**.

1.5.2. **The $\mathsf{U}(1)$-current net $\mathcal{A}_\mathbb{R}$ (revisited).** Let U_1 be the irreducible unitary positive-energy representation of $\mathsf{M\ddot{o}b}$ with lowest weight 1 on a Hilbert space denoted by $\mathcal{H}^1_{\mathcal{A}_\mathbb{R}}$, which can be identified with the one-particle space of the $\mathsf{U}(1)$-current net and remember that there is an embedding $C^\infty(\mathbb{S}^1,\mathbb{R})/\mathbb{R} \to \mathcal{H}^1_{\mathcal{A}_\mathbb{R}}$.

For $I \in \mathcal{I}$ we denote by $H(I) \equiv H_\mathbb{R}(I)$ the local Möbius covariant net of standard subspaces on $\mathcal{H}^1_{\mathcal{A}_\mathbb{R}}$. Like in Section **??** we obtain the $\mathsf{U}(1)$-**current net** $\mathcal{A}_\mathbb{R}$ on $\mathcal{H}_{\mathcal{A}_\mathbb{R}} := \mathrm{e}^{\mathcal{H}^1_{\mathcal{A}_\mathbb{R}}}$ with $\Omega_0 = \mathrm{e}^0$ by defining $\mathcal{A}_\mathbb{R}(I) := R(H(I))$ which is covariant with respect to $U(g) := \mathrm{e}^{U_1(g)}$. For $f \in C^\infty(\mathbb{S}^1,\mathbb{R})$ we consider a self-adjoint operator $J(f)$ given by the generator of the unitary one-parameter group $W(t\cdot f) = \mathrm{e}^{\mathrm{i}t\cdot J(f)}$ with $t \in \mathbb{R}$. This defines the usual current (field operator) smeared with the real test function f, which fulfills $J(f)\Omega_0 = f \in \mathcal{H}^1_{\mathcal{A}_\mathbb{R}}$ and

$$[J(f), J(g)] = 2\mathrm{i}\omega(f,g) = \sum_k k\hat{f}_k\hat{g}_{-k} = \frac{\mathrm{i}}{2\pi}\int f\mathrm{d}g \,.$$

It can be extended to complex test functions via $J(f+\mathrm{i}g) = J(f) + \mathrm{i}J(g)$, and one obtains the usual operator valued (z-picture) distribution $J(z)$ with the relations

$$J(f) = \sum_{n\in\mathbb{Z}} \hat{f}_n J_n = \oint_{\mathbb{S}^1} f(z)J(z)\frac{\mathrm{d}z}{2\pi\mathrm{i}}, \qquad J(z) = \sum_n J_n z^{-n-1}$$

$$[J_m, J_n] = m\delta_{m+n,0}\,,$$

where the modes $J_n = J(e_n)$ with $e_n(\theta) = \mathrm{e}^{\mathrm{i}n\theta}$ satisfy $J_n\Omega_0 = 0$ for $n \geq 0$.

The space $\mathcal{H}_{\mathcal{A}_\mathbb{R}}$ is spanned by vectors of the form $\xi = J_{-n_1}\cdots J_{-n_k}\Omega_0$ with $0 < n_1 \leq \cdots \leq n_k$ with "energy" $N = \sum_m n_m$, i.e. $R(\theta)\xi = \mathrm{e}^{\mathrm{i}N\theta}\xi$. Therefore it is graded with respect to the rotations

$$\mathcal{H}_{\mathcal{A}_\mathbb{R}} = \mathbb{C}\Omega_0 \oplus \bigoplus_{n\in\mathbb{N}} \mathcal{H}_{\mathcal{A}_\mathbb{R},n} \qquad \mathcal{H}_{\mathcal{A}_\mathbb{R},n} = \bigoplus_{k=1}^{n} \bigoplus_{\substack{0<n_1\leq\cdots\leq n_k\\ n_1+\cdots+n_k=n}} \mathbb{C}J_{-n_1}\cdots J_{-n_k}\Omega_0$$

and $\dim\mathcal{H}_{\mathcal{A}_\mathbb{R},n}$ is the number of partitions of n elements, whose generating function $p(t)$ is the inverse of Euler's function $\phi(t) = \prod_{k=1}^\infty(1-t^k)$ and therefore the conformal character of the $\mathsf{U}(1)$-current net is given by ($t = \mathrm{e}^{-\beta}$):

$$\mathrm{tr}_{\mathcal{H}_{\mathcal{A}_\mathbb{R}}}(\mathrm{e}^{-\beta L_0}) = \sum_{n=0}^{\infty} \dim\mathcal{H}_{\mathcal{A}_\mathbb{R},n}\cdot t^n = \prod_{n\in\mathbb{N}}(1-t^n)^{-1}$$

(a conformal character is defined as a formal power series, but it is often convergent for $|t| < 1$ and here we used the formula $(1-z)^{-1} = 1+z+z^2\cdots$). It will be convenient to use the real parametrization $x \in \mathbb{R} \cong \mathbb{S}^1 \setminus \{-1\}$ of the cut circle and use the conventions

$$f(s) = \int_\mathbb{R} \mathrm{e}^{-\mathrm{i}sp}\hat{f}(p)\mathrm{d}p.$$

By writing $f(s) = f_0(\theta(s))$ for $f_0 \in C^\infty(\mathbb{S}^1,\mathbb{R})$ where $\theta(s) = 2\arctan(s)$, the space $\mathcal{H}^1_{\mathcal{A}_\mathbb{R}}$ above can be identified with the space $L^2(\mathbb{R}_+, p\mathrm{d}p)$ in which the space $\mathcal{S}(\mathbb{R},\mathbb{R})$ embeds by restriction of the Fourier transformation to $\mathbb{R}_+$. In other words $\mathcal{H}^1_{\mathcal{A}_\mathbb{R}}$ can be seen as the closure of the space $\mathcal{S}(\mathbb{R},\mathbb{R})$ with complex structure $\mathcal{J}\hat{f}(p) = \mathrm{i}\,\mathrm{sign}(p)\hat{f}(p)$ and the scalar product and sesquilinear form given by:

$$\langle f,g\rangle = \int_{\mathbb{R}_+}\hat{f}(-p)\hat{g}(p)p\mathrm{d}p\,, \qquad \omega(f,g) = \frac{-\mathrm{i}}{2}\int_\mathbb{R}\hat{f}(-p)\hat{g}(p)p\mathrm{d}p = \frac{1}{4\pi}\int_\mathbb{R} f(x)g'(x)\mathrm{d}x\,.$$

Using the above identification we denote for $f \in \mathcal{S}(\mathbb{R},\mathbb{R})$ by $J(f)$ the smeared current with $J(f)\Omega_0 = f \in \mathcal{H}^1_{\mathcal{A}_\mathbb{R}}$. In this parametrization commutation relations read:

$$[J(f),J(g)] = \frac{\mathrm{i}}{2\pi}\int_\mathbb{R} f(x)g'(x)\mathrm{d}x = \int_\mathbb{R}\hat{f}(-p)\hat{g}(p)p\mathrm{d}p\,.$$

1.5.3. **The free complex Fermion net** $\mathrm{Fer}_\mathbb{C}$**.** We construct the net of the free complex Fermion on the circle, which can be seen as the chiral part of the net of the free massless Dirac (or complex) Fermion on two dimensional Minkowski space. The notations of this section are basically in accordance with [Was98], but we use a different convention of positive-energy, which leads to the conjugated complex structure. For giving a simple description of the one-particle space, we consider first the Hilbert space $L^2(\mathbb{S}^1)$ and the Hardy space $H^2(\mathbb{S}^1)$ (see Subsection **??**). It holds that

$$H^2(\mathbb{S}^1) = \{f \in L^2(\mathbb{S}^1) : \hat{f}_n = 0 \text{ for } n < 0\}\,,$$

where $\hat{f}_n$ is the n-th Fourier component of f. We denote the orthogonal projection onto $H^2(\mathbb{S}^1)$ by P.

We remember that the group

$$\mathsf{M\ddot{o}b}^{(2)} \cong \mathrm{SU}(1,1) = \left\{\begin{pmatrix}\alpha & \beta\\ \overline{\beta} & \overline{\alpha}\end{pmatrix} \in \mathsf{GL}(2,\mathbb{C}) : |\alpha|^2 - |\beta|^2 = 1\right\}$$

acts on the circle $\mathbb{S}^1$ by $g\cdot z = \frac{\alpha z+\beta}{\overline{\beta}z+\overline{\alpha}}$. Further there is a unitary action of $\mathsf{SU}(1,1)$ on $L^2(\mathbb{S}^1)$ by

$$(U(g)f)(z) := (V_g f)(z) = \frac{1}{-\overline{\beta}z+\alpha}f(g^{-1}\cdot z)\,.$$

One sees that the projection P commutes with V_g , since $V_g f$ is still an analytic function for $|\alpha| > |\beta|$.

Then one defines a new Hilbert space, the **one-particle space of the free complex Fermi net** by

$$\mathcal{H}^1_{\mathrm{Fer}_\mathbb{C}} = \overline{PL^2(\mathbb{S}^1)} \oplus (1-P)L^2(\mathbb{S}^1)$$

namely, $\mathcal{H}^1_{\mathrm{Fer}_\mathbb{C}}$ is identical with $L^2(\mathbb{S}^1)$ as a real linear space and the multiplication by i is given by $-\mathrm{i}(2P-1)$, or in other words, by $-\mathrm{i}$ on $PL^2(\mathbb{S}^1)$ and i on $(1-P)L^2(\mathbb{S}^1)$. Because P and $U(g)$ commute, the action of SU(1, 1) remains unitary on $\mathcal{H}^1_{\mathrm{Fer}_\mathbb{C}}$.

Then for $I \in \mathcal{I}$ one takes real Hilbert subspaces $K(I) := L^2(I)$ of $\mathcal{H}^1_{\mathrm{Fer}_\mathbb{C}}$. This subspaces turn out to be standard [Was98, Theorem (p. 497)]. If I_1 and I_2 are disjoint intervals, $K(I_1)$ is real orthogonal to $K(I_2)$, in other words $K(I_1) \subset K(I_2)^\perp$, where $K^\perp = \{\xi \in \mathcal{H} : \mathrm{Re}(\xi, K) = 0\}$. It turns out that $I \mapsto K(I)$ is a twisted-local Möbius covariant net of standard subspaces.

The Fermionic second quantization is explained in Section 1.1.2. With this we define the net $\mathrm{Fer}_\mathbb{C}(I) := C(K(I)) = \{a_P(f), a_P(f)^* : f \in L^2(I)\}''$ (where here $a_P(f) := a(\overline{Pf}) + a(P^\perp f)^*$) on $\mathcal{H}_{\mathrm{Fer}_\mathbb{C}} = \Lambda(\mathcal{H}^1_{\mathrm{Fer}_\mathbb{C}}) \cong \Lambda(\overline{PL^2(\mathbb{S}^1)}) \otimes \Lambda(P^\perp L^2(\mathbb{S}^1))$ which is isotonic by definition and fulfills twisted duality, namely by Haag-Araki duality $\mathrm{Fer}_\mathbb{C}(I') = C(K(I)^\perp) = C(K(I))^\perp = \mathrm{Fer}_\mathbb{C}(I)^\perp$. In addition, the net $\mathrm{Fer}_\mathbb{C}$ is Möbius covariant. Indeed, we can take the representation $\Lambda(U(\cdot))$ by promoting the one-particle representation U to the second quantization unitary. It is easy to see that the covariance of this net $\mathrm{Fer}_\mathbb{C}$ follows from the covariance of the net of standard spaces K. The representation $\Lambda(U)$ has positive energy since so does the representation U, and leaves invariant the vacuum vector Ω_0 of the Fock space. Summing up, the net $\mathrm{Fer}_\mathbb{C}$ is a Fermi net (cf. [Was98]). This net is referred to as the **free complex Fermi net** on $\mathbb{S}^1$. The scalar multiplication by a constant phase $\mathrm{e}^{-\mathrm{i}\vartheta}$ in the original structure of the one-particle space is still a unitary operator in the new structure. Its promotion by the second quantization $V(\theta)$ implements an action of U(1) on $\mathrm{Fer}_\mathbb{C}$ by inner symmetry. This will be referred to as the U(1)-gauge action.

For $r \in \frac{1}{2} + \mathbb{Z}$ let $\psi_r = a_P(e_{-r-\frac{1}{2}})$ and $\bar\psi_r = a_P(e_{r-\frac{1}{2}})^*$ where $e_r \in L^2(\mathbb{S}^1)$ with $e_r(\theta) = \mathrm{e}^{\mathrm{i}\theta r}$. The $\psi_r, \bar\psi_r$ are the modes of the free complex Fermion, namely

$$\begin{aligned}\{\psi_n, \psi_m\} = \{\bar\psi_m, \bar\psi_n\} &= 0\\ \{\bar\psi_n, \psi_m\} &= \delta_{m+n,0}\\ \psi_n^* &= \bar\psi_{-n}\end{aligned}$$

and it holds that $\psi_r\Omega_0 = \bar\psi_r\Omega_0 = 0$ for $r \in \frac{1}{2} + \mathbb{N}_0$. Each of ψ_r or $\bar\psi_r$ has norm 1 following from the commutation relation. We can introduce the usual fields smeared with test functions $f, g \in L^2(\mathbb{S}^1)$ and its operator valued distributions in the z-picture:

$$\Psi(f) = \sum_{r\in\frac{1}{2}+\mathbb{Z}} \hat f_r \Psi_r = \oint_{\mathbb{S}^1} f(z) z^{-\frac{1}{2}} \Psi(z) \frac{\mathrm{d}z}{2\pi\mathrm{i}}\,, \qquad \Psi(z) = \sum_{r\in\frac{1}{2}+\mathbb{Z}} \Psi_r z^{-r-\frac{1}{2}}\,,$$

$$\bar\psi(f) = \psi(\overline{f})^* = a_P(e_{-\frac{1}{2}} f)^*\,, \qquad \{\bar\psi(f), \psi(g)\} = \oint f(z) g(z) \frac{\mathrm{d}z}{2\pi\mathrm{i}z}\,,$$

where Ψ is either ψ or $\bar\psi$. The fields $\psi, \bar\psi$ are covariant, e.g. $U(g)\psi(f)U(g)^* = \psi(f_g)$ with

$$f_g(z) = \frac{1}{|\alpha - \overline{\beta} z|} f(g^{-1}z) \quad \text{for} \quad g = \begin{pmatrix}\alpha & \beta\\ \overline{\beta} & \overline{\alpha}\end{pmatrix} \in \mathrm{SU}(1,1)\,.$$

We note that vectors of the form

$$\xi = \psi_{-r_1} \cdots \psi_{-r_k} \bar\psi_{-s_1} \cdots \bar\psi_{-s_\ell} \Omega_0\,, \tag{1.1}$$

with $0 < r_1 < \cdots < r_k$ and $0 < s_1 < \cdots < s_\ell$ form a basis of $\mathcal{H}_{\mathrm{Fer}_\mathbb{C}} = \Lambda(\mathcal{H}^1_{\mathrm{Fer}_\mathbb{C}})$ and that (1.1) is an eigenvector for the rotations, $R(\theta)\xi \equiv \mathrm{e}^{\mathrm{i}\theta L_0}\xi = \mathrm{e}^{\mathrm{i}N\theta}\xi$ with $N = \sum_{j=1}^k r_j + \sum_{j=1}^\ell s_j$ and of the gauge action $V(\theta)\xi \equiv \mathrm{e}^{\mathrm{i}\theta Q}\xi = \mathrm{e}^{\mathrm{i}(k-\ell)\theta}\xi$. In each vector of this basis the r-th energy level can either be empty, be occupied by ψ_{-r} or $\bar\psi_{-r}$ or occupied by both. The contribution of this level to the character $\mathrm{tr}_{\mathcal{H}_{\mathrm{Fer}_\mathbb{C}}}(\mathrm{e}^{-\beta L_0 - EQ})$ is then 1, zt^r, $z^{-1}t^r$ or t^{2r}, respectively, where $t = \mathrm{e}^{-\beta}$ and $z = \mathrm{e}^{-E}$. By summing over all possibilities one gets that the character of $\mathrm{Fer}_\mathbb{C}$ is given by (cf. [Kac98, Reh]):

$$\begin{aligned}\mathrm{tr}_{\mathcal{H}_{\mathrm{Fer}_\mathbb{C}}}(\mathrm{e}^{-\beta L_0 - EQ}) = \mathrm{tr}_{\mathcal{H}_{\mathrm{Fer}_\mathbb{C}}}(t^{L_0}z^Q) &= \prod_{r\in\mathbb{N}_0+\frac12}(1 + zt^r + z^{-1}t^r + t^{2r})\\ &= \prod_{r\in\mathbb{N}_0+\frac12}(1+zt^r)(1+z^{-1}t^r)\\ &= p(t)\sum_{q\in\mathbb{Z}} z^q t^{\frac{q^2}{2}}\,,\end{aligned}$$

where the last equality follows directly from the Jacobi triple product formula (see [Apo76, Theorem 14.6])

$$\prod_{r\in\mathbb{N}}(1+zw^{2r-1})(1+z^{-1}w^{2r-1})(1-w^{2r}) = \sum_{q\in\mathbb{Z}} z^q w^q$$

by setting $2r - 1 = 2n$ and $t = w^2$. In particular, for the local net $\mathrm{Fer}_\mathbb{C}^{\mathsf{U}(1)}$ the character is given by $\mathrm{tr}_{\mathcal{H}^{\mathsf{U}(1)}_{\mathrm{Fer}_\mathbb{C}}}(\mathrm{e}^{-\beta L_0}) = p(t)$, since it is the fixed point with respect to the $\mathsf{U}(1)$-gauge action and the conformal character is the coefficient of z^0.

1.5.4. **The double construction and real Fermion.** Let us define $\mathcal{H}_{\frac12}$ by taking the real Hilbert space obtained by the closure of the antiperiodic functions

$$\{f : \mathbb{R}\to\mathbb{R} : f(\theta+2\pi) = -f(\theta)\} \subset C^\infty(\mathbb{R},\mathbb{R})$$

under the norm

$$\|f\|^2 = 2\sum_{r\in\mathbb{N}_0+\frac12}|\hat f_r|^2$$

where the function is expanded as:

$$f(\theta) = \sum_{r\in\mathbb{Z}+\frac12}\hat f_r \mathrm{e}^{\mathrm{i}r\theta}\,, \qquad\qquad \hat f_{-r} = \overline{\hat f_r}\,.$$

An orthogonal basis of this space is given by

$$\left\{c_r : \theta\mapsto\cos(r\theta), s_r : \theta\mapsto\sin(r\theta) : r\in\mathbb{N}_0+\tfrac12\right\}.$$

This space gets a complex Hilbert space denoted $\mathcal{H}_{\frac12}$ with the complex structure $\mathcal{J} : \hat f_k \mapsto -\mathrm{i}\,\mathrm{sign}(k)\hat f_k$, which acts on the basis by

$$\mathcal{J}c_r = s_r\,, \qquad\qquad \mathcal{J}s_r = -c_r\,, \qquad\qquad r\in\mathbb{N}_0+\frac12$$

and it has an unitary action of the double cover of the Möbius group $\mathsf{M\ddot ob}^{(2)} \cong \mathsf{SU}(1,1)$ by

$$(U(g)f)(z) = \frac{1}{|\alpha - \overline{\beta}z|}f(g^{-1}z) \qquad\qquad g = \begin{pmatrix}\alpha & \beta\\ \bar\beta & \alpha\end{pmatrix}\in\mathsf{SU}(1,1)\,. \tag{1.2}$$

There is an alternative way to describe the space $\mathcal{H}_{\frac{1}{2}}$ starting with a complex structure and basis projection as described in Section **??**. Therefore we start with $(L^2(\mathbb{S}^1),\Gamma)$ with the complex conjugation $\Gamma f = \overline{f}$ and the action of $\mathsf{M\ddot{o}b}_2$ given by the same formula[1] (1.2). We define the basis projection

$$P_{\mathrm{NS}} = \sum_{r\in\mathbb{N}_0+\frac{1}{2}} |e_{-r}\rangle\langle e_{-r}|$$

and get an isorphism between the Hilbert space $PL^2(\mathbb{S}^1)$ and $\mathcal{H}_{\frac{1}{2}}$ by

$$e_{-r} \longmapsto 1/\sqrt{2}(c_r - \mathcal{J}s_r) = \sqrt{2}c_r\,, \qquad (r\in\mathbb{N}_0+\tfrac{1}{2})$$

where $e_r(\mathrm{e}^{\mathrm{i}\theta}) = \mathrm{e}^{r\mathrm{i}\theta}$ with $\theta\in(-\pi,\pi)$. This exactly the correspondence between $(L^2(\mathbb{S}^1),\Gamma,P_{\mathrm{NS}})$ and $\mathcal{H}_{\frac{1}{2}}$ given by Proposition **??**. A net of standard subspaces can e.g. be defined by $K(I) = \operatorname{Re} L^2(I) \subset \operatorname{Re} L^2(\mathbb{S}^1) \cong PL^2(\mathbb{S}^1)$ or by the closure of anti-periodic real functions with support in I, or abstractly using modular localization. This net fulfills twisted Haag duality, i.e. $K(I') = \mathrm{i}K(I)'$.

Proposition 1.5.2 (Double construction)**.** *We can identify the one particle space $\mathcal{H}^1 = L^2(\mathbb{S}^1)_{1-P}$ of $\mathrm{Fer}_{\mathbb{C}}$ with $\mathcal{H}_{\frac{1}{2}}\oplus\mathcal{H}_{\frac{1}{2}}$ as representations of $\mathsf{M\ddot{o}b}^{(2)}$. It can be choosen such that the standard subspace $K(I)$ is identified $K(I)\oplus K(I)$. The $\mathsf{U}(1)$ action on $\mathcal{H}_{\frac{1}{2}}\oplus\mathcal{H}_{\frac{1}{2}}$*

$$V(\vartheta) = \begin{pmatrix}\cos\vartheta & -\sin\vartheta\\ \sin\vartheta & \cos\vartheta\end{pmatrix} \qquad \vartheta\in\mathbb{R}/2\pi\mathbb{Z}\cong\mathsf{U}(1)$$

is given by the multiplication by $\mathrm{e}^{\mathrm{i}_{L^2(\mathbb{S}^1)}\vartheta} = \mathrm{e}^{\mathrm{i}(1-2P)}$ on $\mathcal{H}^1$.

It is convenient to write formally $\mathcal{H}_{\frac{1}{2}}\oplus\mathrm{i}\mathcal{H}_{\frac{1}{2}}$, which reflects that the sum of two real Fermions is a complex Fermion. Then the identification is up to a character of the rotations, which makes the antiperiodic function periodic, given by the obvious identification.

Proof. We can identify as real Hilbert spaces

$$\begin{aligned} M : \mathcal{H}_{\mathbb{R}}\oplus\mathrm{i}\mathcal{H}_{\mathbb{R}} &\to L^2(\mathbb{S}^1)\\ f\oplus\mathrm{i}g &\mapsto e_{-\frac{1}{2}}(f+\mathrm{i}g)\end{aligned}$$

namely $\|c_r\oplus\mathrm{i}s_r\| \equiv \|c_r\|^2 + \|s_r^2\| = 1 = \|e_{r-\frac{1}{2}}\| = \|M(c_r\oplus\mathrm{i}s_r)\|$ and as complex spaces $M(\mathcal{H}_{\mathbb{R}}\oplus i\mathcal{H}_{\mathbb{R}}) = \mathcal{H}^1$, namely for example for $r>0$

$$\begin{aligned} M(\mathcal{J}\oplus\mathcal{J})(c_r\oplus\pm\mathrm{i}s_r) &= M(s_r\oplus\mp\mathrm{i}c_r)\\ &= e_{-\frac{1}{2}}(s_r\mp\mathrm{i}c_r)\\ &= \mp\mathrm{i}e_{\pm r-\frac{1}{2}}\\ &= \mathcal{J}e_{\pm r-\frac{1}{2}}\\ &= \mathcal{J}M(c_r\oplus\pm\mathrm{i}s_r)\,.\end{aligned}$$

[1] We should see this as a priori as a projective representation up to a factor ±1 which we can make into a true representation.

It is $MV_g \oplus V_g = V_g M$, namely

$$\begin{aligned}(M(V_g \oplus V_g)(f \oplus \mathrm{i}h))(z) &= z^{-\frac{1}{2}} \frac{1}{|\alpha - \overline{\beta}z|}(f + \mathrm{i}h)(g^{-1}.z) \\ &= \frac{1}{(\alpha - \overline{\beta}z)^{\frac{1}{2}}(\overline{\alpha}z - \beta)^{\frac{1}{2}}}(f + \mathrm{i}h)\left(\frac{\overline{\alpha}z - \beta}{-\overline{\beta} + \alpha z}\right) \\ &= \frac{1}{\alpha - \overline{\beta}z}\left(\frac{\overline{\alpha}z - \beta}{-\overline{\beta} + \alpha z}\right)^{-\frac{1}{2}}(f + ih)\left(\frac{\overline{\alpha}z - \beta}{-\overline{\beta} + \alpha z}\right) \\ &= \frac{1}{\alpha - \overline{\beta}z} M(f \oplus \mathrm{i}h)(g^{-1}.z) \\ &= V_g M(f \oplus \mathrm{i}h)(z)\,. \end{aligned}$$

It is easy to check that $MV(\theta) = \mathrm{e}^{\mathrm{i}\theta}M$ with i the imaginary unit from $L^2(\mathbb{S}^1)$. Finally if $f \oplus \mathrm{i}g$ has support on I if and only if $M(f \oplus \mathrm{i}g)$ has support in I. □

We can define the net of a real Fermion simply by $\mathrm{Fer}_{\mathbb{R}}(I) := C(K_{\frac{1}{2}}(I))$ on $\Lambda(\mathcal{H}_{\frac{1}{2}})$. Then it holds: $\mathrm{Fer}_{\mathbb{C}} \cong \mathrm{Fer}_{\mathbb{R}} \hat{\otimes} \mathrm{Fer}_{\mathbb{R}}$, where $\hat{\otimes}$ is the twisted tensor product.

1.5.5. **$\mathsf{U}(1)$-current net as a subnet of** $\mathrm{Fer}_{\mathbb{C}}$**.** In this section we use the well-known fact that the Wick product $:\bar{\psi}\psi:$ of the complex Fermion ψ equals the $\mathsf{U}(1)$-current and give an analogue of the Boson–Fermion correspondence (see e.g. [Kac98, 5.2]) in the operator algebraic setting. Let us denote by $\mathcal{D}_0$ the subspace of $\Lambda(\mathcal{H}_P^1)$ of vectors with finite energy:

$$\mathcal{D}_0 := \operatorname{span}\left\{\psi_{-r_1}\cdots\psi_{-r_k}\bar{\psi}_{-s_1}\cdots\bar{\psi}_{-s_l}\Omega_0 : k,l \in \mathbb{N}_0, r_i, s_j \in \mathbb{N} + \frac{1}{2}\right\}.$$

Then we define the unbounded operators on the domain $\mathcal{D}_0$:

$$\begin{aligned} J_n = \sum_{r+s=n} :\bar{\psi}_r\psi_s: \quad &= \quad \sum_{r<0}\bar{\psi}_r\psi_{n-r} - \sum_{r>0}\psi_{n-r}\bar{\psi}_r \\ &= \quad \sum_r (\bar{\psi}_r\psi_{n-r} - (\Omega_0, \bar{\psi}_r\psi_{n-r}\Omega_0)) \end{aligned}$$

with $r, s \in \frac{1}{2} + \mathbb{Z}$. Note that any vector in $\mathcal{D}_0$ is annihilated by ψ_r for sufficiently large r, thus the action of J_n on such a vector can be defined and remains in $\mathcal{D}_0$. In particular, we have $J_n\Omega_0 = 0$ for $n \in \mathbb{N}_0$.

Lemma 1.5.3. *On $\mathcal{D}_0$ it holds that*

(1) $[J_n, \psi_k] = -\psi_{n+k}$ *and* $[J_n, \bar{\psi}_k] = \bar{\psi}_{n+k}$,
(2) $[J_m, J_n] = m\delta_{m+n,0}$.

Proof. Using $[ab, c] = a\{b, c\} - \{a, c\}b$, one obtains $[\bar{\psi}_r\psi_n, \psi_k] = -\delta_{r+k,0}\psi_n$ and $[\psi_n\bar{\psi}_r, \psi_k] = \delta_{r+k,0}\psi_n$ from which directly follows $[J_n, \psi_k] = \sum_{r<0}[\bar{\psi}_r\psi_{n-r}, \psi_k] - \sum_{r>0}[\psi_{n-m}\bar{\psi}_r, \psi_k] = -\psi_{n+k}$. Analogously one shows $[J_n, \bar{\psi}_k] = \bar{\psi}_{n+k}$.

From the Jacobi identity, it follows immediately that $[J_n, J_m]$ commutes with all ψ_k and $\bar{\psi}_k$ and hence $[J_n, J_m]$ is a multiple of the identity, therefore $[J_n, J_m] = (\Omega_0, [J_n, J_m]\Omega_0) \cdot 1$. It is

$$\begin{aligned}[J_n, J_p] &= \sum_{r<0}[J_n, \bar{\psi}_r\psi_{p-r}] - \sum_{r>0}[J_n, \psi_{p-r}\bar{\psi}_r] \\ &= -\sum_{r<0}\left(\bar{\psi}_r\psi_{p-r+n} - \bar{\psi}_{r+n}\psi_{p-r}\right) - \sum_{r>0}\left(\psi_{p-r}\bar{\psi}_{r+n} - \psi_{p-r+n}\bar{\psi}_r\right)\end{aligned}$$

and in the case $p \neq -n$ we get $(\Omega_0, [J_n, J_p]\Omega_0) = 0$, and otherwise

$$(\Omega_0, [J_n, J_{-n}]\Omega_0) = \begin{cases} \sum_{r<0}(\Omega_0, \bar{\psi}_{r+n}\psi_{-r-n}\Omega_0) = \sum_{r=\frac{1}{2}}^{n-\frac{1}{2}}(\Omega_0, \{\bar{\psi}_r, \psi_{-r}\}\Omega_0) & n > 0 \\ -\sum_{r>0}(\Omega_0, \psi_{-r-n}\bar{\psi}_{r+n}\Omega_0) = -\sum_{r=\frac{1}{2}}^{-n-\frac{1}{2}}(\Omega_0, \{\psi_r, \bar{\psi}_{-r}\}\Omega_0) & n < 0 \end{cases}$$
$$= n\,,$$

which completes the proof. □

Let L_0 be the generator of the rotation: $R(\theta) = \mathrm{e}^{\mathrm{i}\theta L_0}$. From its action (see the end of Section 1.5.3) one verifies that $\mathcal{D}_0$ is a core for L_0.

Lemma 1.5.4 (Linear energy bounds)**.** *It holds that $[L_0, J_n] = -nJ_n$ on $\mathcal{D}_0$. For a trigonometric polynomial $f = \sum_n \hat{f}_n e_n$ where the sum is finite and $\xi \in \mathcal{D}_0$, we have*

$$\|J(f)\xi\| \leq c_f \|(L_0 + 1)\xi\|$$
$$\|[L_0, J(f)]\xi\| \leq c_{\partial_\theta f} \|(L_0 + 1)\xi\|,$$

where c_f depends only on f.

Proof. For the commutation relation, it is enough to choose an energy eigenvector $\xi \in \mathcal{D}_0$, i.e. $L_0\xi = N\xi$. It is $J_n L_0 \xi = N J_n \xi$ and

$$L_0 J_n \xi = L_0 \left(\sum_{r<0} \bar{\psi}_r \psi_{n-r} \xi - \sum_{r>0} \psi_{n-r} \bar{\psi}_r \xi \right) = (N - n) J_n \xi,$$

and the first statement follows.

We have seen that ψ_r and $\bar{\psi}_r$ have norm 1 (in Section 1.5.3). First we claim that $\|J_n\xi\| \leq \|(2(L_0 + 1) + |n|)\xi\|$. Let ξ be again an eigenvector of L_0, i.e. $L_0\xi = N\xi$. From the defining sum of J_n, one sees that only $2N + |n| + 2$ terms contribute to $J_n\xi$. Hence we have $\|J_n\xi\| \leq (2N + |n| + 2)\|\xi\| = \|2(L_0 + 1) + |n|\xi\|$. If the inequality holds for eigenvectors, then for $\{\xi_r\}$ with different eigenvalues, we have $\xi_r \perp \xi_s$ and $J_n\xi_r \perp J_n\xi_s$, and hence

$$\begin{aligned} \left\| J_n \sum_r \xi_r \right\|^2 &= \sum_r \|J_n \xi_r\|^2 \\ &\leq \sum_r \|(2(L_0 + 1) + |n|)\xi_r\|^2 \\ &= \left\| (2(L_0 + 1) + |n|) \sum_r \xi_r \right\|^2 \end{aligned}$$

and the general case follows.

For a smeared field, we have

$$\|J(f)\xi\| = \left\| \sum_n \hat{f}_n J_n \xi \right\| \leq 2\tilde{c}_f \|(L_0 + 1)\xi\| + \tilde{c}_{\partial_\theta f} \|\xi\| \leq (2\tilde{c}_f + \tilde{c}_{\partial_\theta f}) \|(L_0 + 1)\xi\|,$$

where $\tilde{c}_f = \sum_n |\hat{f}_n|$. By defining $c_f = 2\tilde{c}_f + \tilde{c}_{\partial_\theta f}$, we obtain the first inequality of the statement. The rest follows by noting that $[L_0, J(f)] = J(\mathrm{i}\partial_\theta f)$. □

For a smooth function $f = \sum_{n\in\mathbb{Z}} \hat{f}_n e_n \in C^\infty(\mathbb{S}^1)$, its Fourier coefficients $\hat{f}_n$ are strongly decreasing and, in particular, it is summable: $\sum_n |\hat{f}_n| = \tilde{c}_f < \infty$. Hence we can naturally extend the definition of the smeared current to smooth functions using the above estimate by

$$J(f) = \sum_{n\in\mathbb{Z}} f_n J_n = \sum_{r,s\in\frac{1}{2}+\mathbb{Z}} f_{r+s} :\psi_r\bar{\psi}_s:,$$

and the same inequality as in Lemma 1.5.4 holds. The operator is closable since we have $J(f) \subset J(\overline{f})^*$ and we still denote the closure by $J(f)$. We note that from the above definition it follows that $J(f)$ is obtained by a limit $\sum_n :\psi(h_n)\bar{\psi}(k_n):$ with suitable functions such that $\sum_n h_n(\theta)k_n(\vartheta) \to 2\pi f(\theta)\delta(\theta-\vartheta)$. This implies covariance of the "field", i.e. $U(g)J(f)U(g)^* = J(f\circ g^{-1})$.

Recall that $\|\psi_r\| = 1$, hence the smeared field is still bounded: $\|\psi(g)\| \le \tilde{c}_g$. We claim that, for $f, g \in C^\infty(\mathbb{S}^1)$ and $\xi \in \mathcal{D}_0$: the vector $\psi(g)\xi$ is in the domain of $J(f)$. Indeed, for a trigonometric polynomial g, we have the estimate

$$\begin{aligned}\|J(f)\psi(g)\xi\| &\le c_f\|(L_0+1)\psi(g)\xi\| \\ &\le c_f(\tilde{c}_g\|\xi\| + \|[L_0,\psi(g)]\xi + \psi(g)L_0\xi\|) \\ &\le c_f(\tilde{c}_g(\|\xi\| + \|L_0\xi\|) + \tilde{c}_{\partial_\theta g}\|\xi\|)\,.\end{aligned}$$

Then if we have a sequence of trigonometric polynomials g_n converging to a smooth function $g \in C^\infty(\mathbb{S}^1)$, the sequence $\{J(f)\psi(g_n)\xi\}$ is also converging.

Lemma 1.5.5. *For $\xi, \eta \in \mathcal{D}_0$, it holds that*

$$\begin{aligned}&[J(f),\psi(g)]\xi = -\psi(f\cdot g)\xi \\ &[J(f),\bar{\psi}(g)]\xi = \bar{\psi}(f\cdot g)\xi \\ &(J(\bar{f})\xi, J(g)\eta) = (J(\bar{g})\xi, J(f)\eta) + 2\mathrm{i}\omega(f,g)(\xi,\eta).\end{aligned}$$

Proof. For trigonometric polynomials f, g, the statements can be proved easily from Lemma 1.5.3. The general case is shown by approximating first f by polynomials, then g, according to the convergence considered above (as for the third statement, obviously the order of limits does not matter). □

We need the following well-known result [DF77, Theorem 3.1]:

Theorem 1.5.6 (The commutator theorem)**.** *Let H be a positive self-adjoint operator and A, B symmetric operators defined on a core $\mathcal{D}_0$ for $(H+1)^2$. Assume that there is a constant C such that*

$$\begin{gathered}\|A\xi\| \le C\|(H+1)\xi\|, \quad \|B\xi\| \le C\|(H+1)\xi\|, \\ \|[H,A]\xi\| \le C\|(H+1)\xi\|, \quad \|[H,B]\xi\| \le C\|(H+1)\xi\|, \\ (A\xi, B\eta) = (B\xi, A\eta) \text{ for any } \xi,\eta \in \mathcal{D}_0.\end{gathered}$$

Then A and B are essentially self-adjoint on any core of H and any bounded functional calculus of A and B commute.

Remark 1.5.7. In the original literature [DF77], this Theorem is proved under the assumption of certain operator inequalities. In fact, what is really used in the proof of commutativity of bounded functions is the norm estimates $\|A(H+1)^{-1}\| < C$, $\|[H,A](H+1)^{-1}\| < C$ etc. and they follow from the assumptions here. The essential self-adjointness of A and B can be proved by [RS75, Theorem X.37]. An analogous application of this theorem with norm estimates can be found in [BSM90].

By the commutator theorem, we get that $J(f)$ is self-adjoint for $f \in C^\infty(\mathbb{S}^1, \mathbb{R})$ and that all bounded functions of $J(f)$ commute with all bounded functions of $J(g)$ for $f, g \in C^\infty(\mathbb{S}^1, \mathbb{R})$ with disjoint support.

Let I be a proper interval and let us define the von Neumann algebra

$$\mathcal{B}(I) = \{e^{iJ(f)} : \operatorname{supp} f \subset I\}''.$$

The local net $\mathcal{B}(I)$ restricted to $\overline{\mathcal{B}(I)\Omega_0}$ can be identified with the $\mathsf{U}(1)$-current net $\mathcal{A}_\mathbb{R}$ on $\mathcal{H}_{\mathcal{A}_\mathbb{R}}$, in particular we can identify $\overline{\mathcal{B}(I)\Omega_0} \cong \mathcal{H}_{\mathcal{A}_\mathbb{R}}$.

Proposition 1.5.8. *Let I be a proper interval, then $\mathcal{B}(I) \subset \mathrm{Fer}_\mathbb{C}^{\mathsf{U}(1)}(I)$.*

Proof. We see that $\mathcal{B}(I)$ commutes with $\mathrm{Fer}_\mathbb{C}(I') = \{c(g) : g \in L^2(I')\}''$ because, for f, g with disjoint supports, $c(g)$ commutes with $J(f)$ on a core by Lemma 1.5.5 and therefore any spectral projection of $c(g)$ commutes with $J(f)$, and hence with any bounded functions of $J(f)$.

Further because $J(f)$ commutes by construction with the gauge action $V(t)$ and is in particular even because $V(\pi) = \Gamma$, it follows that $\mathcal{B}(I)$ lies in the twisted commutant $\mathrm{Fer}_\mathbb{C}(I')^\perp$. By twisted Haag duality it is $\mathcal{B}(I) \subset \mathrm{Fer}_\mathbb{C}(I')^\perp = \mathrm{Fer}_\mathbb{C}(I)$ and therefore $\mathcal{B}(I) = \mathcal{B}(I)^{\mathsf{U}(1)} \subset \mathrm{Fer}_\mathbb{C}^{\mathsf{U}(1)}(I)$. □

Since the covariance has been seen, we have the following:

Corollary 1.5.9. *$\mathcal{B}$ is a subnet of* $\mathrm{Fer}_\mathbb{C}^{\mathsf{U}(1)}$.

Now the following is straightforward.

Proposition 1.5.10 (Algebraic version of Boson–Fermion Correspondence)**.** *The $\mathsf{U}(1)$-fixed point subnet of the complex free Fermion net $\mathrm{Fer}_\mathbb{C}$ is the $\mathsf{U}(1)$-current net, i.e.* $\mathrm{Fer}_\mathbb{C}^{\mathsf{U}(1)} = \mathcal{B} \cong \mathcal{A}_\mathbb{R}$.

Proof. Let us see $\mathcal{B}$ as a subnet of the Fermi net $\mathrm{Fer}_\mathbb{C}^{\mathsf{U}(1)}$ on $\mathcal{H}_{\mathrm{Fer}_\mathbb{C}}^{\mathsf{U}(1)} \equiv \mathcal{H}_{\cdot,0}$. Further $\overline{\mathcal{B}(I)\Omega}$ does not depend on I by the same proof of the Reeh-Schlieder property and is clearly a subspace of $\mathcal{H}_{\mathrm{Fer}_\mathbb{C}}^{\mathsf{U}(1)} \equiv \mathcal{H}_{\cdot,0}$.

In fact they coincide, since we have confirmed that $\mathrm{tr}_{\mathcal{H}_{\mathcal{A}_\mathbb{R}}}(e^{-\beta L_0}) = \mathrm{tr}_{\mathcal{H}_{\cdot,0}}(e^{-\beta L_0}) = p(t)$, where $e^{-\beta} = t$, namely, their conformal characters coincide (see also Section **??**). □

Remark 1.5.11. For G a group we denote by $\mathsf{L}G = C^\infty(\mathbb{S}^1, G)$ the **loop group**, which forms a group by pointwise multiplication.

- The net $\mathcal{A}_\mathbb{R}$ can be seen as a quantization of $\mathsf{L}\mathbb{R}$, seen as an additive group.
- The net $\mathrm{Fer}_\mathbb{C}$ can be seen as a quantization of $\mathsf{LU}(1)$. It lives on a bigger Hilbert space because $\mathsf{LU}(1)$ naturally contains $\mathsf{L}\mathbb{R}$ as a subgroup but it contains more elements with non-trivial winding number. The $\mathsf{U}(1)$ action is exactly the grading with respect to the winding number and the fixed point is therefore $\mathsf{L}\mathbb{R}$.

We could have used loop groups to prove the Boson–Fermion correspondence. But instead we made direct contact with fields and Fourier modes and learnt how to exponentiate these back to nets.

One of the main difficulties with the subject is that there are many different pictures. Note that we try to define things by intrinsic properties, for example the $\mathsf{U}(1)$-current net can be defined to be the second quantization net of the lowest weight $\ell = 1$ representation of $\mathsf{M\ddot{o}b}$.

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