LECTURE NOTES: ALGEBRAIC AND TOPOLOGICAL QUANTUM FIELD THEORY
CHAPTER 1: QUANTUM MECHANICS

MARCEL BISCHOFF

ABSTRACT. ATTENTION: not proof read lecture notes in progress.....

CONTENTS

—Lecture 15—

$$\mathfrak{A} = \overline{\bigcup_{O\in\mathcal{K}} \mathcal{A}(O)}^{\|\cdot\|} \tag{1.1}$$

TODO

—Lecture 16—

Proposition 1.0.1. *The category Δ has direct sums.*

Proof. Let $\rho_i \in \Delta(O_i)$ for $i = 1,2$. Let p be a projection in $\mathcal{A}(O_1)$. Chooese $O \supset \overline{O_1 \cup O_2}$, then by property B there are isometries $v_1, v_2 \in \mathcal{A}(O)$, such that $v_i^* v_i = 1$, $v_1 v_1^* = e$ and $v_2 v_2^* = 1 - e$. Then this form Cuntz algebra $\mathcal{O}_2$:

$$\sum_{i=1}^{2} v_i v_i^* = 1, \qquad v_i^* v_j = \delta_{i,j} \tag{1.2}$$

inside $\mathcal{A}(O)$.

Define $\rho \in \mathrm{End}(\mathfrak{A})$

$$\rho(x) = v_1\rho_1(x)v_1^* + v_2\rho_2(x)v_2^*, \qquad x \in \mathfrak{A} \tag{1.3}$$

which is a unital endomorphism due to (1.2).

Since $v_i \in \mathcal{A}(O)$ and ρ_i are localized in O_i, ρ is localized in O.

We claim ρ is transportable. Choose $\tilde{O}$. Because ρ_i is transportable, there are $\tilde{\rho}_i$ and $u_i \in \mathrm{Hom}(\rho_i, \tilde{\rho}_i)$. As before choose $\tilde{v}_i$ (for example by assuming that $\tilde{\rho}_i$ are actually localized in $\tilde{O}_i$ with $\overline{\tilde{O}_i} \subset \tilde{O}$ and define

$$\tilde{\rho}(x) = \tilde{v}_1\tilde{\rho}_1(x)\tilde{v}_1^* + \tilde{v}_2\tilde{\rho}_2(x)\tilde{v}_2^*, \qquad x \in \mathfrak{A}\,, \tag{1.4}$$

which is localized in $\tilde{O}$.

Date: today.

It is immediate that $\tilde{v}_i u_i v_i^* \in \mathrm{Hom}(\rho, \tilde{\rho})$. Then

$$w = \sum_i v_i u_i v_i^* \tag{1.5}$$

$$w^* w = \sum_{i,j} (v_i u_i^* \tilde{v}_i^*)(\tilde{v}_i u_i v_i^*) \tag{1.6}$$

$$= \sum_i v_i^* v_i = 1 \tag{1.7}$$

$$w w^* = \sum_{i,j} (\tilde{v}_i u_i v_i^*)(v_i u_i^* \tilde{v}_i^*) \tag{1.8}$$

$$= \sum_i \tilde{v}_i^* \tilde{v}_i = 1\,, \tag{1.9}$$

thus $w \in \mathrm{Hom}(\rho, \tilde{\rho})$ is a unitary. □

We denote $[\rho] = \{U\rho(\,\cdot\,)U^* : U \in \mathcal{A}(O) \text{ unitary}\}$ which is called the sector. We note that for sectors the direct sum is well-defined:

$$[\rho_1] \oplus [\rho_2] := [\rho] \tag{1.10}$$

where ρ is as in the proof above. The same way we can define:

$$\bigoplus_{i=1}^{n} [\rho_i]\,, \qquad\qquad N[\rho] := \bigoplus_{i=1}^{N} [\rho]\,, \quad N \in \mathbb{N}\,. \tag{1.11}$$

Proposition 1.0.2. *The category Δ has subobjects*

Proof. Let $\rho \in \Delta(O)$ and $p \in \mathrm{Hom}(\rho, \rho)$. By Haag duality $p \in \mathcal{A}(O)$. Choose $\tilde{O} \supset \overline{O}$. By property B there is an isometry $v \in \mathcal{A}(O_1)$, such that $vv^* = p$ and $v^* v = 1$ and define

$$\tilde{\rho}(x) = v^* \rho(x) v \qquad\qquad x \in \mathfrak{A} \tag{1.12}$$

Then $v \in \mathrm{Hom}(\tilde{\rho}, \rho)$:

$$v\tilde{\rho}(x) = p\rho(x)v = \rho(x)pv = \rho(x)v\,. \tag{1.13}$$

$\tilde{\rho}$ is transportable. Suppose $\mathcal{O}_2 \subset \overline{\mathcal{O}}_3$ doubles cone. Since ρ is transportable there is ρ_3 localized in O_3 and a unitary $u \in \mathrm{Hom}(\rho, \rho_3)$. Then $p_3 = upu^* \in \mathrm{Hom}(\rho_3, \rho_3)$. By property B there is a $v_2 \in \mathcal{A}(\tilde{O}_2)$, such that $v_2 v_2^* = p_2$. Let $\tilde{\sigma} = v_2 \rho_3 v_2^*$ which is localized in O_2 and $w = v_2^* uv \in \mathrm{Hom}(\tilde{\rho}, \tilde{\sigma})$.

Then w is unitary

$$w^* w = v^* u^* v_2 v_2^* uv = v^* u^* p_3 uv = v^* pv = v^* vv^* v = 1 \tag{1.14}$$

and similarly $ww^* = 1$. □

Definition 1.0.3. If $\mathcal{C}$ is a $\mathbb{C}$-linear category, then an object ρ is called **irreducible** if $\mathrm{Hom}(\rho, \rho) = \mathbb{C} \cdot 1_\rho$.

We note that $\mathrm{id} := \mathrm{id}_{\mathfrak{A}}$ is irreducible since the vacuum representation is assumed to be irreducible.

Proposition 1.0.4. *Let $\rho \in \Delta(O_1)$ and $\sigma \in \Delta(O_2)$, then $\rho \circ \sigma \in \Delta(O)$ for every $O \subset O_1 \cup O_2$.*

Proof. It follows immediately that $\rho\sigma$ is localized in O. We show that it is transportable. Choose $\tilde{O}$, then exists $\tilde{\rho}, \tilde{\sigma}$ localized in $\tilde{O}$ and unitaries $u \in \mathrm{Hom}(\rho, \tilde{\rho})$ and $v \in \mathrm{Hom}(\sigma, \tilde{\sigma})$. Define the unitary $w = u\rho(v)$, then

$$w\rho\sigma(x) = u\rho(v)\rho(\sigma(x)) = u\rho(v\sigma(x)) = u\rho(\tilde{\sigma}(x)v) = u\rho(\tilde{\sigma}(x))\rho(v) = \tilde{\rho}(\tilde{\sigma}(x))u\rho(v) = \tilde{\rho}\tilde{\sigma}(x)w \tag{1.15}$$

thus $w \in \mathrm{Hom}(\rho\sigma, \tilde{\rho}\tilde{\sigma})$. □

This defines a (monoidal product) $\otimes$-product:

$$\otimes\colon \text{Obj}(\Delta) \times \text{Obj}(\Delta) \to \text{Obj}(\Delta) \qquad (\rho, \sigma) \mapsto \rho \otimes \sigma = \rho\sigma \tag{1.16}$$

Proposition 1.0.5. *Let $S \in \text{Hom}(\rho, \tilde{\rho})$ and $T \in \text{Hom}(\sigma, \tilde{\sigma})$, then $S\rho(T) \in \text{Hom}(\rho\sigma, \tilde{\rho}\tilde{\sigma})$.*

Proof.

$$S\rho(T)\rho \circ \sigma(x) = S\rho(T\sigma(x)) \tag{1.17}$$

$$= S\rho(\tilde{\sigma}(x)T) \tag{1.18}$$

$$= S\rho(\tilde{\sigma}(x))\rho(T) \tag{1.19}$$

$$= \tilde{\rho} \circ \tilde{\sigma}(x)S\rho(T) \tag{1.20}$$

□

We define the bilinear map:

$$\times\colon \text{Hom}(\rho, \tilde{\rho}) \times \text{Hom}(\sigma, \tilde{\sigma}) \longrightarrow \text{Hom}(\rho\sigma, \tilde{\rho}\tilde{\sigma}) \tag{1.21}$$

$$(S, T) \longmapsto S \times T := S\rho(T) \tag{1.22}$$

Remark 1.0.6. We have

$$S\rho(T) = \tilde{\rho}(T)S \tag{1.23}$$

Proposition 1.0.7. *Let $S_i \in \text{Hom}(\rho_i, \sigma_i)$ and $T_i \in \text{Hom}(\sigma_i, \tau_i)$ for $i = 1, 2$. Then*

$$(T_1 \times T_2)(S_1 \times S_2) = (T_1S_1) \times (T_2S_2) \tag{1.24}$$

$$(S_1 \times T_1)^* = S_1^* \times T_1^* \tag{1.25}$$

Proof. Straight forward calculation. □

—String diagrams—

The identity id $\in \text{End}(\mathfrak{A})$ is a tensor unit and we have natural associativity of the tensor product, i.e.

$$\text{id} \otimes \rho = \rho = \rho \otimes \text{id} \tag{1.26}$$

$$1_{\text{id}} \times T = T = T \times 1_{\text{id}} \tag{1.27}$$

$$\rho_1 \otimes (\rho_2 \otimes \rho_3) = (\rho_1 \otimes \rho_2) \otimes \rho_3 \tag{1.28}$$

$$T_1 \times (T_2 \times T_3) = (T_1 \times T_2) \times T_3 \tag{1.29}$$

References