LECTURE NOTES: ALGEBRAIC AND TOPOLOGICAL QUANTUM FIELD THEORY CHAPTER 4: SUPERSELECTION THEORY IN LOW DIMENSIONS

MARCEL BISCHOFF

Abstract. ATTENTION: not proof read lecture notes in progress.....

Contents

—Lecture: Pointed braided categories—
—Lecture: Representation of Free Boson—
—Lecture: Extension of Free Boson and Nets associated with even Lattices—

1.1. Modular Tensor Categories

Let $\mathcal{C}$ be Δ_f be the category of localized representations of a net $\mathcal{A}$ or $\mathcal{C}$ be a braided fusion category of $\mathrm{End}(N)$ for a type III factor N.

We assume that $\mathcal{C}$ has finitely many irreducible sectors: $\mathrm{Irr}(\mathcal{C}) = \{[\rho_1 = \mathrm{id}], \ldots, [\rho_n]\}$, for example $\mathcal{C} \cong \mathrm{Rep}(G)$ for G a finite group. We already classified the case when $d_k := d\rho_k$ are all equal to one, namely $\mathrm{Irr}(\mathcal{C})$ form an abelian group this group together with the twists $\omega_k = \omega_{\rho_k}$ form a complete invariant. We get an example from a local net on the circle for every even lattice L. Namely, the localized endomorphisms of $\mathcal{A}_L$ are characterized by the abelian group $A = L^*/L$ and $\omega_q = \exp(\mathrm{i}\pi\langle q, q\rangle)$ for all $q \in L^*/L$.

Since $\mathcal{C}$ is closed under the tensor product we the fusion rule coefficients $N_{ij}^k \in \mathbb{N}_0$ by decomposition:

$$[\rho_i\rho_j] = \bigoplus_k N_{ij}^k[\rho_k].$$

In other words, $N_{ij}^k = \dim \mathrm{Hom}(\rho_i\rho_j, \rho_k)$. Using the solutions of the conjugate equation we get isomorphism of the vector spaces:

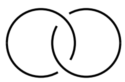
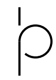
$$\mathrm{Hom}(\rho_i\rho_j, \rho_k) \cong \mathrm{Hom}(\rho_i, \rho_k\bar\rho_j) \cong \mathrm{Hom}(\rho_j, \bar\rho_i\rho_k)$$

and therefore relations between the fusion rule coefficients called **Frobenius duality**. –picture–

We get an involution on the index set $[\rho_{\bar i}] = [\bar\rho_i]$ and we showed that $N_{i\bar j}^0 = \delta_{j,\bar i}$.

Date: today.

Many informations is encoded in the double twists $\varepsilon(\rho_i,\rho_j)^*\varepsilon(\rho_j,\rho_i)$, for example it is the identity in the symmetric case. In general, this depends on the choice of the representant in $[\rho_i]$ but if we take the categorical trace of it it gets invariant. We define this to be Y_{ij}, it correspond to the i, j-colored Hopf link:

$Y_{i,j} = \bar{\rho}_i$  $\bar{\rho}_j$; $\omega_\lambda \cdot 1_\lambda =$ λ  λ (1.1)

Note if $\mathcal{C}$ is symmetric $Y_{ij} = d_i d_j$ does not contain much information, all rows (colums) are multiple of each other.

Lemma 1.1.1. *If $\mathcal{C}$ is a braided fusion category as above, then:*

(1) $Y_{0i} = Y_{i0} = d_i$
(2) $Y_{ij} = Y_{ji} = \overline{Y_{\bar{i}\bar{j}}} = Y_{\bar{i}\bar{j}}$
(3) $Y_{ij} = \sum_k N_{ij}^k \frac{\omega_i\omega_j}{\omega_k} d_k$
(4) $\frac{1}{d_j} Y_{ij} Y_{kj} = \sum_k N_{ij}^k Y_{mj}$

Corollary 1.1.2. *Define $\chi_i^{(m)} = (Y_{im}/d_m)$, then we have*

$$\chi_i^{(m)}\chi^{(m)}m_j = \sum_l N_{ij}^k \chi_k^{(m)}$$

i.e. the $[\rho_i] \mapsto \chi_i^m$ are characters of the fusion rule algebra $\mathbb{C}[\mathrm{Irr}(\mathcal{C})]$ for every m. Note $\chi_i^{(0)} = d_i$ and $\chi^{(0)}$ is the trivial representation and if $\mathcal{C}$ is symmetric than $\chi^{(m)}$ is the trivial reprensentation for all m.

Lemma 1.1.3. *Let $Y_l = (Y_{lj})_l \in \mathbb{C}^n$ and $\langle\,\cdot\,,\,\cdot\,\rangle$ the standard scalar product in $\mathbb{C}^n$. Then either*

- *Y_l and Y_j are mutually orthogonal, i.e. $\langle Y_l, Y_j\rangle = 0$, or*
- *they are parallel: $d_j Y_l = d_l Y_j$.*

Proof. It is enough to show

$$d_j^{-1} Y_{ij}\langle Y_l, Y_j\rangle = \langle Y_l, N_i Y_j\rangle = d_l^{-1}\langle Y_l, Y_j\rangle$$

□

Definition 1.1.4. We call a sector Y_i parallel to Y_0 **degenerate** ($\chi^{(i)} = \chi^{(0)}$).

Lemma 1.1.5. *A sector $[\rho_i]$ is degenerate if and only if $\varepsilon(\rho_i,\rho_j)\varepsilon(\rho_j,\rho_i) = 1$ for all $[\rho_j] \in \mathrm{Irr}(\mathcal{C})$.*

Corollary 1.1.6. *If $[\rho_i]$ is degenerate, then it has permutation statistics. In particular, $\omega_i = \pm 1$ and $d_i \in \mathbb{N}$.*

Proof. If $\varepsilon(\rho_i,\rho_j)\varepsilon(\rho_j,\rho_i) = 1$ for all $[\rho_j] \in \mathrm{Irr}(\mathcal{C})$, then $Y_{ij} = d_i d_j$ for all j.

If $Y_{ij} = d_i d_j$ for all j, then $\omega_k/\omega_i\omega_j = 1$ if $N_i j^k > 0$ which follows from comparing $Y_{ij} = \sum_k N_{ij}^k \frac{\omega_i\omega_j}{\omega_k} d_k$ with $d_i d_j = \sum_k N_{ij}^k d_k$. . But this implies that $\varepsilon(\rho_i,\rho_j)\varepsilon(\rho_j,\rho_i) = 1$. □

Proposition 1.1.7. *TFAE:*

- $[\rho_0] = [\mathrm{id}]$ *is the only degenerate sector.*
- *The matrix Y is invertible*

- *The number* $\sigma = \sum d_i^2 \omega_i^{-1}$ *satisfies* $|\sigma|^2 = \sum_i d_i^2$. *The matrices:*

$$S := |\sigma|^{-1} Y, \quad T =: \left(\frac{\sigma}{|\sigma|}\right)^{1/3} \mathrm{diag}(w_k)$$

satisfy the Verlinde relations:

$$SS^* = TT^* = 1_N \tag{1.2}$$

$$STSTST = C \tag{1.3}$$

$$S^2 = CTC \qquad = CT = T \tag{1.4}$$

with $C_{ij} = \delta_{i,\bar{j}}$ *the* ***conjugation matrix***.

Corollary 1.1.8. *The matrix* S *diagonalizes the fusion rule matrices* $N_i = (N_{ij}^k)_{ik}$, *i.e. we get the Verlinde formula:*

$$N_{ik}^m = \sum_j \frac{S_{ij} S_{kj} \bar{S}_{mj}}{S_{0j}}\,. \tag{1.5}$$

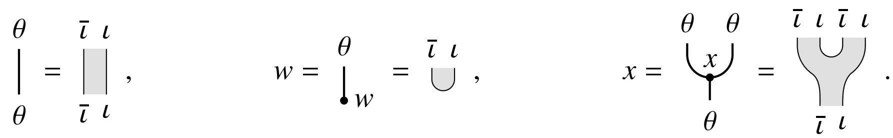
Indeed, this formula can be written as: $N_i = S D_i S^{-1}$ *with* $D_i = \mathrm{diag}(Y_{ij}/d_j)_j = \mathrm{diag}(\chi_i^{(j)})_j$.

1.2. Nets of subfactors and extension

Assume we have an irreducible inclusion $\mathcal{A} \subset \mathcal{B}$ of local nets. This means there is a representation $\{\pi_O \colon \mathcal{A}(O) \to \mathcal{B}(O) \subset \mathrm{B}(\mathcal{H}_\mathcal{B})\}$ and a isometry $V \colon \mathcal{H}_\mathcal{A} \to \mathcal{H}_\mathcal{B}$ intertwining all the structure that we assume (e.g. $VU_\mathcal{A}(g) = U_\mathcal{B}(g)V$, $V\Omega_\mathcal{A} = \Omega_\mathcal{B}$) it follows from the fact that π is unitary equivalend to a localizable and transportable endomorphism ρ with $\rho \in \Delta(\mathcal{A})$. In higher dimension we have seen that there is a compact gauge group (G, k) and the field net is the maximal twisted local irreducible extension. It follows that $\mathcal{A} \subset \mathcal{B} \subset \mathcal{F}^{\mathbb{Z}_2}$ and that $\pi \prec \pi_{\mathrm{reg}}$ completely determines the $\mathcal{B}$ and the local extensions are in one-to-correspondence with closed subgroups of $G/\langle k \rangle$ (G or $G/\mathbb{Z}_2$).

In low dimension most of this breaks down. In general, there is unique no maximal local extension. For a maximal local extdension $\mathcal{B}$ $\mathrm{Rep}(\mathcal{B})$ is in general also non-trivial, but it was recently shown that if $\mathrm{Rep}(\mathcal{A})$ is modular that all maximal local extensions have braided equivalent representation categories.

We fix an interval I. We assume that $A = \mathcal{A}(I) \subset \mathcal{B}(I) = B$ has finite index. One can show that the index does not depend on I. Finite index means that with $\iota \colon A \to B$ there exist a conjugate $\bar\iota \colon B \to A$, and a standard solution to the conjugate equation $v \colon \mathrm{id}_M \to \iota\bar\iota =: \gamma$ and $w \colon \mathrm{id}_N \to \bar\iota\iota = \delta$. In this context γ is called the **canonical endomorphism** and δ the **dual canonical endomorphism**. The dual canonical endomorphism can be extended to a localized endomorphism of $\mathcal{A}$ localized in I. Indeed, it is unitary equivalent to the representation $\{\pi_I \colon \mathcal{A} \to \mathrm{B}(\mathcal{H}_\mathcal{B})\}$. Like discussed above, in general, θ alone does not determine the extension $\mathcal{B}$ because a second cohomology is involved. A complete invariant is given by $(\theta, w, x := \iota(v))$ which was found by Longo and called a **Q-system**.



It is a categorical version of an algebra, namely an algebra object in $\mathrm{Rep}(\mathcal{A})$ like. As comparison: a unital associative algebra is an algebra object in the category of vector spaces. Namely, w is the unit and x^* is the multiplication. From the conjugate equation it easily follows (written this way it is actually a co-algebra):

$$xx = \theta(x)x \qquad (x \otimes 1_\theta)x = (1_\theta \otimes x)x \qquad \text{(associativity)}$$
$$w^*x = \theta(w^*)x = 1_\theta \qquad (w^* \otimes 1_\theta)x = (1_\theta \otimes w^*)x = 1_\theta \qquad \text{(unit law)}.$$

In graphical notation this reads:

and by taking $*$ corresponding to reflect the pictures vertical on gets a algebra.

Definition 1.2.1. A triple $\Theta = (\theta, w, x)$ with $\theta \in \mathrm{End}(N)$ and isometries $\sqrt{d\theta}^{-1} w \colon \mathrm{id}_N \to \theta$ and $\sqrt{d\theta}^{-1} x \colon \theta \to \theta^2$, which we will graphically display as

is called a **Q-system** (cf. [?Lo1994, ?LoRo1997])

Two Q-systems $\Theta = (\theta, w, x)$ and $\tilde{\Theta} = (\tilde{\theta}, \tilde{w}, \tilde{x})$ in $\mathrm{End}(N)$ are called equivalent, if there is a unitary $u \in \mathrm{Hom}(\theta, \tilde{\theta})$, such that

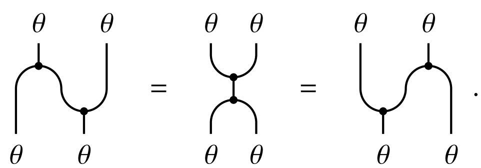
$$\tilde{x}u = (u \otimes u)x \equiv u\theta(u)x\,; \qquad\qquad u\tilde{w} = w$$

hold, or graphically:

A Q-system in a C*-tensor category automatically [?LoRo1997] fulfills the "Frobenius law" which is easy to check for our example from a subfactor:

$$(x^* \otimes 1_\theta)(1_\theta \otimes x) \equiv x^*\theta(x) \quad = \quad xx^* \quad = \quad (1_\theta \otimes x^*)(x \otimes 1_\theta) \equiv \theta(x^*)x$$

or graphically:



Definition 1.2.2. Let $\Theta = (\theta, w, x)$ be a Q-system. It is called **irreducible** if $\dim \mathrm{Hom}(\mathrm{id}, \theta) = 1$ and standard if $(\bar{\theta}, r, \bar{r})$ with $\bar{\theta} = \theta$, $r = \bar{r} = xw$ is a standard solution for the conjugate equation for θ.

So an irreducible finite index subfactor $A \subset B$ of type III gives an irreducible standard Q-system in $\mathrm{End}(A)$.

Conversely, given an abstract Q-system in $\mathrm{End}(A)$ there is an overfactor $B \supset A$ by the Longo-Rehren construction.

Proposition 1.2.3 (Longo–Rehren construction)**.** *Let* $\Theta = (\theta, w, x)$ *be an irreducible standard Q-system in* $\mathrm{End}(A)$ *for a type III factor A. Then there is a factor B with canonical inclusion* $\iota\colon A \to B$ *with conjugate* $\bar{\iota}\colon B \to A$ *and* $v \in \mathrm{Hom}(\mathrm{id}_M, \iota\bar{\iota})$*, such that* $(\bar{\iota}, v, w)$ *fulfills the conjugate equation for* ι *and* $A \subset B$ *is irreducible.*

Proof. The short way is to define B to be the Jones basic construction $E_0(A) \subset A \subset B$ where $E_0 = \sqrt{d\theta}^{-1}x^*\theta(\,\cdot\,)x$ is a conditional expectation by the Q-system properties.

But we can build B directly from A by adjoining one element v fulfilling:

$$v\iota(n) = \iota\theta(n)v$$

with $n \in A$ and $\iota\colon A \to B$ the embedding of A into the still to be constructed algebra B. This means $v \in \mathrm{Hom}(\iota, \iota\theta)$ and its square and adjoint are defined to be

$$v^2 = \iota(x)v \qquad\qquad v^* = \iota(wx^*)v \tag{1.6}$$

A general element of B can be written as $\iota(n)v$ for some $n \in A$. The unit is $1_B = \iota(w^*)v$ and $(\iota(n)v)^* = v^*\iota(n^*) - \iota(w^*x^*\theta(n^*))v$. By the Frobenius property this turns the set $B = \{\iota(n)v : n \in A\}$ into a $*$-algebra. We induce the weak topology from A to B with help of the conditional expectation $E\colon B \to A$ given by

$$E(m) = \sqrt{d\theta}^{-1}w^*\iota(m)w \qquad\qquad E(vv^*) = \sqrt{d\theta}^{-1}1_A \tag{1.7}$$

where $\bar{\iota}\colon B \to A : \iota(n)v \mapsto \theta(n)x$ and (w, v) is a standard solution to the conjugate equation. □

Proposition 1.2.4. *Let $\iota(A) \subset B$ finite index irreducible as above.*

- *There is a anti-isomorphism between* $\mathrm{Hom}(\rho, \theta)$ *and* $\mathrm{Hom}(\iota, \iota\rho)$.

$$\sigma \prec \theta \equiv \bar{\iota}\iota \quad \Longleftrightarrow \quad \exists \psi \in B : \psi\iota(n) = \iota(\sigma(n))\psi \text{ for all } n \in A\,. \tag{1.8}$$

- *Let* $\theta = \sum w_i \rho_i w_i^*$ *with isometries* $w_i \in A$ *(*$\rho_0 = \mathrm{id}$, $w_0 \sim w$*) and* ρ_i *irreducible. Then every element in* b *can be uniquely written as*

$$b = \sum_i \iota(a_i)\psi_i \qquad\qquad a_i \in A \tag{1.9}$$

with isometries $\psi_i \sim \iota(w_i)^* v$.

The first is just Frobenius reciprocity.

Proof. Every element b can be uniquely written as $b = \iota(a)v$ for a unique $a \in A$, therfore:

$$b = \iota(a)v = \sum_i \iota(a w_i w_i^*) = \sum_i \iota(a w_i)\iota(w_i)^* v\,. \tag{1.10}$$

□

Let $\mathcal{A}$ be net. If $\theta \in \Delta_f(O)$, we get an extension $\mathcal{A}(O) \subset \mathcal{B}(O) = \iota(\mathcal{A}(O))v$. This can be extended to the net.

—4intervals—

1.3. Complete Rationality and Modularity

We fix a local Möbius covariant net $\mathcal{A}$ on S^1. We assume the additional properties which together are called **complete rationality**:

Split property: For every pair $I_1, I_2 \in \mathcal{I}$ with $\bar{I}_1 \subset I_2$ there is a type one factor M, such that $\mathcal{A}(I_1) \subset M \subset \mathcal{A}(I_2)$.

Finite μ index: Let $I_1, \ldots, I_4$ divide the circle. Then with

$$\mathcal{A}(E) = \mathcal{A}(I_1) \vee \mathcal{A}(I_3) \qquad \hat{\mathcal{A}}(E) = \mathcal{A}(E')' = (\mathcal{A}(I_2) \vee \mathcal{A}(I_4))' \tag{1.11}$$

the inclusion

$$\mathcal{A}(E) \subset \hat{\mathcal{A}}(E) := \mathcal{A}(E')' \tag{1.12}$$

has finite index $\mu(\mathcal{A}) := [\hat{\mathcal{A}}(E) : \mathcal{A}(E)]$.

Strong additivity: For $I \in \mathcal{I}$ and $p \in I$ and $I_1, I_2 \in \mathcal{I}$ such that $I_1 \cup I_2 = I \setminus \{p\}$ we have

$$\mathcal{A}(I) = \mathcal{A}(I_1) \vee \mathcal{A}(I_2) \tag{1.13}$$

Proposition 1.3.1. *Let $\rho \in \mathrm{Rep}_f(\mathcal{A})$, then there is a conjugate $\bar{\rho} \in \mathrm{Rep}_f(\mathcal{A})$*

Proof. Let $\rho \in \mathrm{Rep}^I_f(\mathcal{A})$ and $\tilde{\rho} \in \mathrm{Rep}^{I'}_f(\mathcal{A})$. By Bisongano-Wichmann property $j(\mathcal{A}(I)) = \mathcal{A}(I')$ with $j = \mathrm{Ad}(J_{(\mathcal{A}(I),\Omega)})$. Define $\bar{\rho} = j\tilde{\rho}j$. Since j acts geometrically this is again localized in I. Since $\rho(\mathcal{A}(I)) \subset \mathcal{A}(I)$ having finite index one can show that ρ and $\bar{\rho}$ are conjugate by showing that $\rho j \tilde{\rho} j$ is Longo's canonical endomorphism for $\rho(\mathcal{A}(I)) \subset \mathcal{A}(I)$. □

Recall that if we have $\iota(N) \subset M$ is finite index iff there is a $\bar{\iota}\colon M \to N$, such that $\mathrm{id}_M \prec \gamma := \iota\bar{\iota}$ and $\mathrm{id}_N \prec \theta := \bar{\iota}\iota$.

From Frobenius reciprocity follows for $\sigma \in \mathrm{End}(N)$

$$\sigma \prec \theta \equiv \bar{\iota}\iota \iff \exists \psi \in M : \psi\iota(n) = \iota(\sigma(n))\psi \text{ for all } n \in N\,. \tag{1.14}$$

Namely, $\psi \in \mathrm{Hom}(\iota, \psi\iota)$ which is antisomorphic to $\mathrm{Hom}(\sigma, \theta)$. If $\theta = \sum \mathrm{Ad}\, u_i\rho_i$ and we choose ψ_i corresponding to u_i, every in $m \in M$ can be uniquely written as:

$$m = \sum \iota(n_i)\psi_i \qquad n_i \in N\,. \tag{1.15}$$

Let $I_1, \cdots, I_4$ be a partition of the circle into intervals and $E = I_1 \cup I_3$, thus $E' = I_2 \cup I_4$. We want to apply (1.14) to

$$\mathcal{A}(E) \subset \hat{\mathcal{A}}(E) := \mathcal{A}(E')'. \tag{1.16}$$

Let $\rho_i \in \mathrm{Rep}^{I_1}_f(\mathcal{A})$ irreducible and $\bar{\rho}_i \in \mathrm{Rep}^{I_3}_f(\mathcal{A})$ a conjugate of ρ_i. Then there is a isometry

$$R_i \in \mathrm{Hom}(\mathrm{id}, \rho_i\bar{\rho}_i) \subset \hat{\mathcal{A}}(E) \tag{1.17}$$

where the inclusion follows by Haag duality and strong additivity.

Thus by (1.14) we have $\rho_i\bar{\rho}_i \prec \theta_E := \bar{\iota}_E\iota_E$ the dual canonical endomorphism of the inclusion . We can take a direct sum over all equivalence classes of irreducible objects in $\mathrm{Rep}_f(\mathcal{A})$ and obtain

$$\bigoplus_i \rho_i\bar{\rho}_i \prec \theta_E \tag{1.18}$$

Proposition 1.3.2. *We have*

$$\sum_i (d\rho_i)^2 = [\hat{\mathcal{A}}(E) : \mathcal{A}(E)] < \infty\,,$$

in particular there are only finitely many irreducibles sectors in $\mathrm{Rep}_f(\mathcal{A})$.

Proof. We have proven $\sum_i (d\rho_i)^2 \leq [\hat{\mathcal{A}}(E) : \mathcal{A}(E)]$.
—TODO— □

Proposition 1.3.3. *Every irreducible representation (on a seperable Hilbert space!) has finite statistical dimension.*

Proof. Given π irreducible, we choose equivalent localized representations $\sigma_i \in \mathrm{Rep}^{I_i}(\mathcal{A})$ for $i = 1, 3$. Then there is a unitary

$$u \in \mathrm{Hom}(\sigma_1, \sigma_3) \subset \hat{\mathcal{A}}(E)$$

Using (1.15) we can write $u = \sum_i u_i R_i$ and for $x \in \mathcal{A}(I_1)$ we have $u_i \rho_i \sigma_1(x) = x u_i$ for all $x \in \mathcal{A}(I_1)$:

$$u\sigma_1(x) = xu = \sum_i x u_i R_i = \sum_i u_i R_i \sigma_1(x) = \sum_i u_i \rho_i \sigma_1(x) R_i$$

For some i we have $u_i \neq 0$. Using split property and the conditional expectation $E_1 \colon \mathcal{A}(E) \to \mathcal{A}(I_1)$ we have $E_1(u_i) \in \mathrm{Hom}(\rho_i \sigma_1, \mathrm{id})$, thus $\sigma_1 \cong \bar{\rho}_i$. But this means that σ has finite statistical dimension. □

The goal is to proof that [id] is the only degenerate sector,i.e. any $\rho \in \mathrm{Rep}_f(\mathcal{A})$ with $\varepsilon(\rho, \sigma) = \varepsilon(\sigma, \rho)^*$ for all σ (irreducible) is a direct sum of the vacuum representation $[\rho] = N[\mathrm{id}]$.

Lemma 1.3.4. *Let ρ be a finite sector localized in I_2 and η irreducible and localized in I_2.*

- **Criterion for degeneracy:** *Let $\eta_{1,3} \cong \eta$ localized in $I_{1,3}$, respectively, and T be the (up to a phase unique) unitary "charge transporter" in* $\mathrm{Hom}(\eta_1, \eta_3)$.
  Then $\rho(\sigma, \eta) = \rho(\eta, \sigma)^*)$ if and only if $\rho(T) = T$ for
- **Criterion for triviality:** *ρ acts trivial on $\hat{\mathcal{A}}(I_{123})$ then (if and only if)* $[\rho]$ *is a multiple of the vacuum representation.*
  Here $I_{123} = S^1 \setminus \bar{I}_4$

Proof. Let $T_i \in \mathrm{Hom}(\rho, \rho_i)$. Assuming clockwise order we have

$$\varepsilon(\rho, \eta) = T_3^* \rho(T_3) \tag{1.19}$$

$$\varepsilon(\eta, \rho) = T_1^* \rho(T_1) \tag{1.20}$$

Then

$$\varepsilon(\rho, \eta)\varepsilon(\eta, \rho) \equiv T_3^* \rho(T_3) \rho(T_1^*) T_1 = 1$$

if and only if $\rho(T_3 T_1^*) = T_3 T_1^*$. But $T_3 T_1^*$ equals T up to a phase and the statement follows. □

Proof. Let $I_2 \Subset J \Subset I_{123} = S^1 \setminus \bar{I}_4$. By the split property there are type I factors M_a and M_b, such that

$$\mathcal{A}(I_2) \subset M_a \subset \mathcal{A}(J) \subset M_b \subset \mathcal{A}(I_{123})$$

Since $\mathcal{A}(I_2) \subset M_1$ and ρ is acting trivial on $M_1' \cup \hat{\mathcal{A}}(I_{13}) \subset \mathcal{A}(I_2)' \cap \hat{\mathcal{A}}(I_{123})$ and

$$\begin{aligned}\rho(M_1) &\subset (M_1' \cap \tilde{\mathcal{A}}(I_{123})' \cap M_2 \\ &\subset (M_1' \cap M_2)' \cap M_2 \\ &\subset M_1\end{aligned}$$

thus $\rho \restriction M_1 \in \mathrm{End}(M_1)$ which implies by the type I property that $\rho \restriction M_1 = \sum_{i=1}^N \mathrm{Ad}\, v_i$ for $\{v_i\}_{i=1}^N$ a Cuntz algebra $\mathcal{O}_N$ in $(\mathcal{A}(I_2)' \cap M_a)' \cap M_a = \mathcal{A}(I_2)$. Since $\rho \restriction \mathcal{A}(I_2) = \eta \restriction \mathcal{A}(I_2)$ and $\rho \restriction \mathcal{A}(I_{13}) = \mathrm{id} = \eta \restriction (\mathcal{A}(I_{13})$ and strong aditivity follows that $[\rho] = N[\mathrm{id}]$. □

Proposition 1.3.5. $\mathrm{Rep}_f(\mathcal{A})$ *is non-degenerate and thus a UMTC.*

Proof. Let ρ be localized in I_2 and be degenerate. Then for $\{\rho_i^{(1,3)}\}$ localized in $I_{1,3}$, respectively, and $T_i \operatorname{Hom}(\rho_i^{(1)}, \rho_i^{(3)})$ it follows $\rho(T_i) = T_i$ by the criterion of degeneracy.

Since

$$\hat{\mathcal{A}}(I_{13}) = \mathcal{A}(I_{13}) \vee \left\{R_i \colon \operatorname{id} \to \rho_i^{(1)}\bar{\rho}_i^{(3)}\right\}''$$

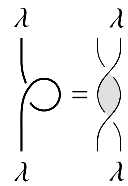
and R_i equals up to a phase $T_i^*\bar{r}_i \colon \operatorname{id} \to \rho_i^{(3)}\bar{\rho}_i^{(3)} \to \rho_i^{(1)}\bar{\rho}_i^{(3)}$ with $\bar{r}_i \in \operatorname{Hom}(\operatorname{id}, \rho_i^{(3)}\bar{\rho}_i^{(3)}) \subset \mathcal{A}(I_3)$.

Now ρ acts trivial on $\mathcal{A}(I_{13})$ but also on T_i, so it acts trivial on R_i and therefore on on $\hat{\mathcal{A}}(I_{13})$. The criterion for triviality gives $[\rho] = N[\operatorname{id}]$. So the only irreducible degenerate sector is [id] which proves the statement. □

1.4. From Modular Tensor Categories to 3-Manifold invariant

1.4.1. **Invariants of direct links/ribbons.** Let $\mathcal{C}$ be a ribbon category.

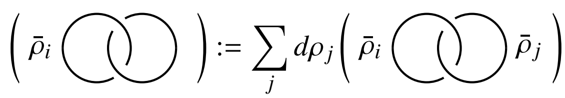
We can avoid arrows by assuming that they always point upwards where labelled and we can draw ribbons by lines using the **blackboard framing**, that means we deform the ribbon in a way that we can lie it flat on a surface and that the same side is always on top. For example a twist of the ribbon we replace by what we called a twist of the string:



The graphical calculus provides and invariant of directed ribbons colored by elements in $\mathcal{C}$. In particular, to every link of directed ribbons with each component colored by an object in $\mathcal{C}$ we can associate a number.

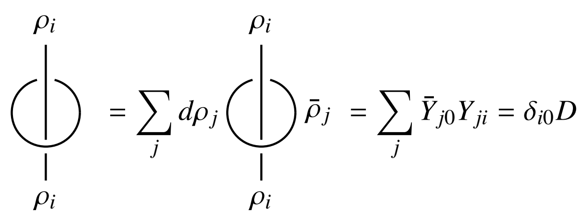
1.4.2. **An important relation in UMTCs.** We assume now that $\mathcal{C}$ is modular.

We use the convention that if a ribbon link has no label, then we sum over Irr($\mathcal{C}$) weighed by the dimension, for example



(1.21)

Opening the ρ_i ring in the picture we get:



(1.22)

—The killing ring

—

$p_\pm$

1.4.3. **Invariants of 3-manifolds.** Let M_1 and $M)2$ be two manifolds of the same dimension. Let N_1 be a component of ∂M_1 and N_2 a component of ∂M_2 and $f\colon N_1 \to N_2$ an orientation preserving homeomorphism.

Define

$$M_1 \cup_f M_2 := (M_1 \sqcup M_2)/\{(x, y) : y = f(x) \text{ for } x \in N_1\}$$

Then $M_f = M_1 \cup_f M_2$ is againg a manifold and we say that M_f is obtained by glueing M_1 and M_2 using f.

If $f' = f \circ \varphi$ for some homeomorphism $\varphi\colon N_1 \xrightarrow{\sim} N_1$ which extends to $M_1 \xrightarrow{\sim} M_1$, then $M_f \simeq M_{f'}$.

M_f does only depend on the isotopy class of f, i.e. it does not chang when we continuously deform f.

Kurby moves

The invariant