Pointed Unitary Fusion Categories

We have a finite group $G \to \mathrm{Out}(M)$. Then $\alpha_g \alpha_h = \mathrm{Ad}(u_{g,h})\alpha_{gh}$ for some $u\colon G \times G \to \mathrm{U}(M)$.

$$\alpha_g \alpha_h \alpha_k = \mathrm{Ad}(u_{g,h})\alpha_{gh}\alpha_k = \mathrm{Ad}(u_{g,h}u_{gh,k})\alpha_{ghk}$$
$$\alpha_g \alpha_h \alpha_k = \alpha_g(\mathrm{Ad}(u_{h,k}))\alpha_g \alpha_{hk} = \mathrm{Ad}(\alpha_g(u_{h,k})u_{g,hk})\alpha_{ghk}$$

and thus there is an $\omega\colon G \times G \times G \to \mathbb{T}$, such that

$$u_{g,h}u_{gh,k} = \omega(g,h,k)\alpha_g(u_{h,k})u_{g,hk}$$

which is a cocycle:

$$\begin{aligned}[d\omega](g_1, g_2, g_3, g_4) \\ = \omega(g_2, g_3, g_4)\omega(g_1 g_2, g_3, g_4)^{-1} \\ \omega(g_1, g_2 g_3, g_4)\omega(g_1, g_2, g_3 g_4)^{-1}\omega(g_1, g_2, g_3) = 1\end{aligned}$$

$$\omega(g_1, g_2, g_3)\omega(g_1, g_2 g_3, g_4)\omega(g_2, g_3, g_4) = \omega(g_1 g_2, g_3, g_4)\omega(g_1, g_2, g_3 g_4)$$

Pointed Braided Unitary Fusion Categories

Let $\omega \in H^3(G, \mathbb{T})$ and $\Omega\colon G \times G \to \mathbb{T}$, such that:

$$\omega(g_2, g_3, g_1)^{-1}\Omega(g_1, g_2g_3)\omega(g_1, g_2, g_3)^{-1} = \Omega(g_1, g_3)\omega(g_2, g_1, g_3)\Omega(g_1, g_2)$$
$$\omega(g_3, g_1, g_2)\Omega(g_1g_2, g_3)\omega(g_1, g_2, g_3) = \Omega(g_1, g_3)\omega(g_1, g_3, g_2)\Omega(g_2, g_3)$$

Then we denote the abelian group of all such (ω, Ω) by $Z^3_{\mathrm{ab}}(G, \mathbb{T})$. The coboundaries are (b, B) with

$$b(g_1, g_2, g_3) = h(g_2, g_3)h(g_1g_2, g_3)^{-1}h(h_1, g_2g_3)h(g_1, g_2)$$
$$B(g_1, g_2) = h(g, g_2)h(g_2, g_1)^{-1}$$

for some $h\colon G \times G \to \mathbb{T}$. Denote this by $B^3_{\mathrm{ab}}(G, \mathbb{T})$ and

$$H^3_{\mathrm{ab}}(G, \mathbb{T}) = \frac{Z^3_{\mathrm{ab}}(G, \mathbb{T})}{B^3_{\mathrm{ab}}(G, \mathbb{T})}$$

Theorem

The map

$$\begin{aligned} H^3_{\mathrm{ab}}(G,\mathbb{T}) &\longrightarrow \{q\colon G\to\mathbb{T}\} \\ (\omega,\Omega) &\longmapsto q \text{ with } q(a)=\Omega(a,a) \end{aligned}$$

is an injective homorphism of abelian groups.

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Proof.

It can be directly computed that it is an homorphism. To see that the kernel is trivial note that $q\equiv 1$ implies that our category is symmetric and by Doplicher–Roberts theorem it is equivalent to $\mathrm{Rep}(\hat{G})$ and thus ω must have been trivial. □

Therefore, for abelian fusion rules, i.e. all $d\rho = 1$ the category Δ_f is completely characterized by the abelian group describing the fusion rules and the twists on the group.