MODULARITY OF THE CATEGORY OF REPRESENTATION OF A CONFORMAL NET, II

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Abstract. Notes from the "Conformal Field Theory and Operator Algebras workshop," August 2010, Oregon.

Outline:

(1) Introduction
(2) Two interval inclusions
(3) Modularity

Goal. Let $\mathcal{A}$ be a completely rational conformal net. Orit showed the first few of these:

(1) **Semisimplicity:** Every seperable non-degenerate rep is completely reducible.
(2) The number of unitary equiv. classes of irreducible reps is finite
(3) **Finite statistics:** Every separable irreducible representation has finite statistical dimension
(4) **Modularity:** $\mathrm{Rep}_f(\mathcal{A})$ has a monoid structure with simple unit and duals (conjugates) and a maximally non-degenerate braiding, thus is modular.

1. Introduction

Assume $\mathcal{A}$ is a completely rational conformal net, i.e.

$$\mathcal{I} \ni I \longmapsto \mathcal{A}(I) \subset B(H_0)$$

with H_0 the vacuum Hilbert space, $\Omega \in H_0$ the vacuum vector, $U \curvearrowright H_0$ unitary positive energy representation of $PSU(1,1)$. These data fullfil some axioms (Corbett) plus the additional assumption of *complete rationality*:

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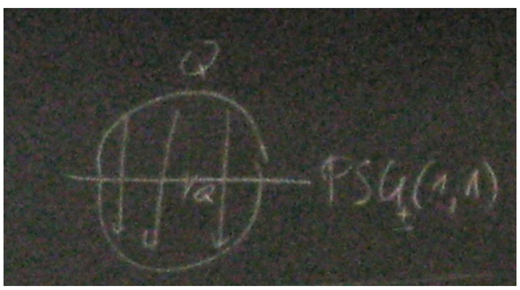
Available online at `http://math.mit.edu/~eep/CFTworkshop`. Please email `eep@math.mit.edu` with corrections and improvements!

(1) strong additivity
(2) split property
(3) finite μ_2 index

Recall a representation of $\mathcal{A}$ is a collection of reps $\{\pi_I\}_{I\in\mathcal{I}}$ with $\pi_I : \mathcal{A}(I) \to B(H)$ which are compatible. If H separable (then we call π a seperable representation), for all $I \in \mathcal{I}$ there is $\rho \simeq \pi$ (we also write $\rho \in [\pi]$; the equivalence class $[\pi]$ is called sector) on H_0 with $\rho_{I'} = id_{A(I')}$. Thus the representation acts trivial outside I. ρ then is called *localized in* I. One has a monoidal structure, given by composition of localized endomorphism (Yoh showed relation to Connes fusion).

Conjugates: Let $\pi \simeq \rho$ be a separable non-degenerate representation localized in I. Let P, Q be two other intervals.

Let $r_Q \in PSU_{\pm}(1,1)$ reflection associated to the intervall Q, cf:



Then we can define another representation by

$$\bar{\rho_I}(x) = J_P \rho_{r_Q(I)}(J_Q x J_Q) J_P$$

where J_P is the modular conjugation for the algbra $\mathcal{A}(P)$. i.e. $J_P\mathcal{A}(P)J_P = \mathcal{A}(P)'$. Remember that we have Bisognano-Wichman property, telling us that $J_P x J_P = U(r_P)xU(r_P)^*$ holds, where U is now the extended (anti) unitary representation of $PSU_{\pm}(1,1)$, i.e. J_P acts geometrically by a reflection. This ensures the above formular is well defined.

It turns out the equivalence class $[\bar{\rho}_I]$ does not depend on P, Q.

Theorem 1.1. *If π is separable and irreducible with finite statistical dimension, then there exists a conjugate representation $\bar{\pi}$. If π is Möbius covariant, then also $\bar{\pi}$. In particular if $\rho \in [\pi]$ like above then $\bar{\rho} \in [\bar{\pi}]$*

So the conjugate representation is given by the above formular up to some choice in the unitary equivalence class.

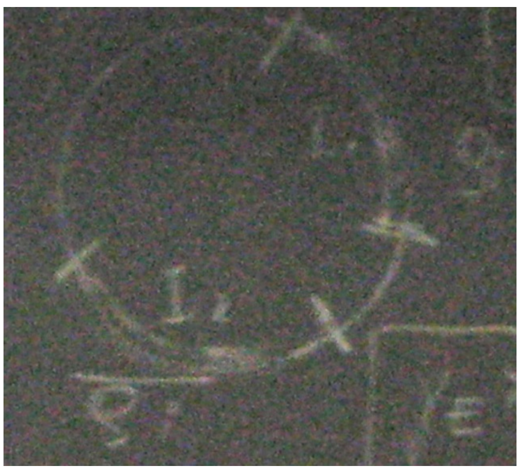
2. Two interval inclusions

We begin with some fact from subfactor theory

Fact. Let $N \subset M$ be an inclusion of type III factors, which is irreducible (ie $N' \cap M = \mathbb{C}\mathbf{1}$) and has finite index: $[M : N] \leq \infty$. We assume we have a canonical endomorphism $\gamma : M \hookrightarrow N$, $\gamma(x) = J_N J_M x J_M J_N$ for $x \in M$. Then are equivalent:

(1) $\sigma \in \mathrm{End}(N) : \sigma \prec \gamma|_N$, i.e. there is $U \in N$ such that $U\sigma(x) = \gamma(x)U$
(2) There is $\psi \in M$ such that $\psi x = \sigma(x)\psi$ for all $x \in N$.

This we want to apply to the two intervall inclusion $\mathcal{A}(E) \subset \hat{\mathcal{A}}(E) := \mathcal{A}(E')'$



with the canonical endomorphism $\gamma_E : \hat{\mathcal{A}}(E) \hookrightarrow \mathcal{A}(E)$.

Pick π_i an irreducible separable representation with finite index, $\rho_i \in [\pi_i]$ localized on I_1.

Then exist a conjugate $\bar{\pi}_i$ and we pick $\bar{\rho}_i \in [\bar{\pi}_i]$ localized in I_2.

There exist a up to constant unique intertwiner (think of co-evaluation map) $R_i \in \mathrm{Hom}(\mathbf{1}, \rho_i\bar{\rho}_i) \in \mathcal{A}(E)$, i.e. $R_i(x) = \rho_i(\bar{\rho}_i(x))R_i$.

Thus using $\sigma = \rho_i\bar{\rho}_i$ in the above fact we get $\rho_i\bar{\rho}_i \prec \lambda_E = \gamma_E|_{\mathcal{A}(E)}$. On the lefthand side we can even take a sum over mutually non-equivalent representations with finite index Γ_f and the inequivality still holds:

$$\bigoplus_{i \in \Gamma_f} \rho_i\bar{\rho}_i \prec \lambda_E = \gamma_E|_{\mathcal{A}(E)}$$

because the endomorphism are mutually inequivalent. It turns out by some further arguments:

$$\bigoplus_{i\in\Gamma_f} \rho_i\bar{\rho}_i \simeq \lambda_E = \gamma_E|_{\mathcal{A}(E)}$$

Taking the index on both sides one can conclude:

$$\sum_{\Gamma_f} d(\rho_i)^2 = [\hat{\mathcal{A}}(E) : \mathcal{A}(E)] = \mu_2$$

We will use another fact from subfactor theory

Fact. Let $\gamma(x) = \sum_i U_i\sigma_i(x)U_i^*$ for $x \in N$ with σ_i irreducible, U_i partial isometries, such that $\sum_i U_i^*U_i = \mathbf{1}$, $U_jU_i^* = \delta_{ij}\mathbf{1}$. Then every $x \in M$ is of the form $x = \sum x_i\psi_i$ for unique $x_i \in N$.

So, for each $x \in \hat{\mathcal{A}}(E)$ we have a decomposition $x = \sum_{i\in\Gamma_f} x_iR_i$ with unique $x_i \in \mathcal{A}(E)$. Thus every element of the bigger factor can be written as elements of the smaller subfactor and intertwiner $\{R_i\}$:

$$\hat{\mathcal{A}}(E) = \mathcal{A}(E) \vee \{R_i\}'$$

The two-intervall inclusion is connected to the intertwiner R_i, thus connected to the representation theory of the net.

3. Modularity

Proposition 3.1. *Every irreducible seperable representation of $\mathcal{A}$ has finite statistical dimension.*

Proof. Sketch: Let $\rho, \rho' \in [\pi]$ be localized in the two components of E respectivly and $u \in \mathrm{Hom}(\rho, \rho') \subset \hat{\mathcal{A}}(E)$ their intertwiner. By the last fact we can uniquely write u as $u = \sum u_iR_i$. Then exist an i such that $u_i \neq 0$ and a short calculation shows that $u_i \in \mathrm{Hom}(\rho_i\rho, id)$, i.e. there exist an non trivial intertwiner $\rho_i\rho$ with the vacuum representation for some i. Duality implies the existence of a non-trivial intertwiner between ρ and $\bar{\rho}_i$ given essentially by:

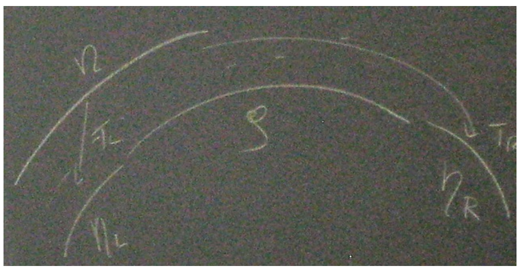
$$\rho \xrightarrow{coev_{\bar{\rho}_i}\otimes 1} \bar{\rho}_i\rho_i\rho \xrightarrow{1\otimes u_i} \bar{\rho}_i$$

and because $\rho, \bar{\rho}_i$ both are irreducible this means $\rho \simeq \bar{\rho}_i$. □

Next: what's the braiding in this category? Braiding is given by a bijective morphism $\epsilon(\rho,\eta) \in \mathrm{Hom}(\rho\eta,\eta\rho)$ satisfying some identities.

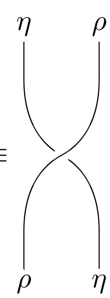
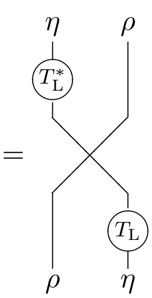
The idea how to define ϵ is to transport ρ and η in disjoint regions (so they commute), exchange the order, and than transport back. This does not depend one the explicit choice of the regions. One could for example transport η to the left or to the right, this gives in particular two (a priori) inequivalent choices.

So let ρ, η be localized in some intervalls, cf

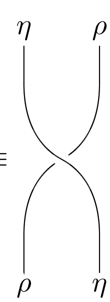
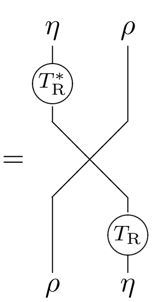


Let $\eta_{L/R} \in [\eta]$ be to equivalent representations localized left and right from ρ, respectively and $T_{L/R} \in \mathrm{Hom}(\eta,\eta_{L/R})$ intertwiners. Note that $\rho\eta_{R/L} = \eta_{R/L}\rho$.

Define $\epsilon(\rho,\eta)$

$\epsilon(\rho,\eta) \equiv$  :=  $= T_{\mathrm{L}}^*\rho(T_{\mathrm{L}})$

Then

$\epsilon(\eta,\rho)^* \equiv$  :=  $= T_{\mathrm{R}}^*\rho(T_{\mathrm{R}})$

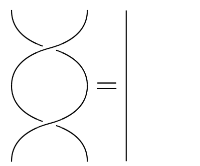
thus is given by the other choice.

Note: $T^*_{\mathrm{L/R}}\rho(T_{\mathrm{L/R}})$ is indeed

$$\rho\eta \xrightarrow{1\otimes T_{L/R}} \rho\eta_{N/L} = \eta_{N/L}\rho \xrightarrow{T^*_{L/R}\otimes 1} \bar{\rho}_i$$

using that the categorical tensorproduct $\rho\eta \equiv \rho \otimes \eta$ is the composition of localized endomorphism.

Definition. ρ and η have *trivial monodromy* if $\epsilon(\rho, \eta) = \epsilon(\eta, \rho)^*$ or equivalently $\epsilon_M(\rho, \eta) := \epsilon(\rho, \eta)\epsilon(\eta, \rho) = \mathbf{1}$, i.e.



Note that $\epsilon_M([\rho], [\eta]) = \epsilon_M(\rho, \eta)$ is well-defined, i.e. the monodromy just depends on sectors and not on the representations itself.

Definition. π separable, non-degenerate representation of $\mathcal{A}$ is called *finite* if one of the following equivalent conditions holds

- π is a finite direct sum of irreps.
- π has finite statistical dimension
- $\pi(C^*(\mathcal{A}))'$ is finite.

Let $\mathrm{Rep}_f(\mathcal{A})$ be the category of all finite reps.

Definition. ρ is called degenerate with respect to braiding if $\epsilon_M(\rho, \eta) = 1$ for all $\eta \in \mathrm{Rep}_f(\mathcal{A})$.

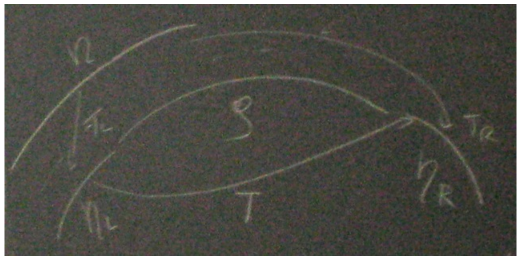
The center $Z_2(\mathrm{Rep}_f)$ is the set of degenerate w.r.t. braiding reps.

Note: in a modular category $\mathcal{C}$, $Z_2(\mathcal{C})$ is trivial, i.e sums of $\mathbf{1}$. This is the most non-trivial fact to check.

We use two ingredients:

Criterion for degeneracy: $\epsilon_M(\rho, \eta) = 1$ iff $\rho(T) = T$ for $T \in \mathrm{Hom}(\eta_L, \eta_R)$.

Proof. $\epsilon_M(\rho_\eta) \equiv T_L^*\rho(T_LT_R^*)T_R = 1$ iff $\rho(T_LT_R^*) = T_LT_R^*$. The statement follows, realizing $T_LT_R^*$ equals T^* up to some constant:



∎

Criterion for triviality of a representation: If ρ act trivially on $\hat{\mathcal{A}}(E)$ then $\rho \simeq N \cdot id$, thus trivial.

Theorem 3.1. *$Z_2(\mathrm{Rep}_f \mathcal{A})$ is trivial thus $\mathrm{Rep}_f \mathcal{A}$ is modular.*

Proof. $\pi \in Z_2(\mathrm{Rep}_f(\mathcal{A}))$ and $\rho \in [\pi]$ localized as above and E the union of intervalls left and right from the localization intervall of ρ. $\rho \in Z_2$ implies $\rho(T) = \mathbf{1}$ for all possible charge transporters T from left to the right using the first criterion.

We have seen that the big factor $\hat{\mathcal{A}}(E)$ is generated by the small $\mathcal{A}(E)$ and the intertwinner R_i, this turns out to be equivalent with $\hat{\mathcal{A}}(E)$ generated by $\mathcal{A}(E)$ and interwiner T_i which transport $\eta = \rho_i$ from left to right, i.e.

$$\hat{\mathcal{A}}(E) = \mathcal{A}(E) \vee \{R_i\} = \mathcal{A}(E) \vee \{T_i\}$$

By definition ρ acts trivially on $\mathcal{A}(E)$, but also on all charge transporters T_i thus on $\hat{\mathcal{A}}(E)$. But this is the second criteria which implies triviality of ρ thus π. Thus the center is trivial. ∎