Local nets on Minkowski half-plane associated to lattices

Marcel Bischoff

Department of Mathematics
University of Rome "Tor Vergata"

Bucharest, April 29th, 2011

- Algebraic quantum field theory: A family of algebras containing all local observables associated to space-time regions.
- Many structural results, recently also construction of interesting models
- Conformal field theory (CFT) in 1 and 2 dimension described by AQFT quite successful, e.g. partial classification results (e.g. $c < 1$) (Kawahigashi and Longo, 2004)
- Boundary Conformal Quantum Field Theory (BCFT) on Minkowski half-plane: (Longo and Rehren, 2004)
- Boundary Quantum Field Theory (BQFT) on Minkowski half-plane: (Longo and Witten, 2010)

$\mathcal{H}$ complex Hilbert space, $H \subset \mathcal{H}$ real subspace. Symplectic complement:

$$H' = \{x \in \mathcal{H} : \mathrm{Im}(x, H) = 0\} = \mathrm{i}H^{\perp}$$

Standard subspace: closed, real subspace $H \subset \mathcal{H}$ with $\overline{H + \mathrm{i}H} = \mathcal{H}$ and $H \cap \mathrm{i}H = \{0\}$.

Define antilinear unbounded closed involutive ($S^2 \subset 1$) operator

$$S_H : x + \mathrm{i}y \mapsto x - \mathrm{i}y \text{ for } x, y \in H.$$

Conversely S densely defined closed, antilinear involution on $\mathcal{H}$, $H_S = \{x \in \mathcal{H} : Sx = x\}$ is a standard subspace:

standard subspaces H	$\overset{1:1}{\longleftrightarrow}$	densely defined, closed, antilinear involutions S

Modular Theory: Polar decomposition $S_H = J_H \Delta_H^{1/2}$

$$J_H H = H' \quad \Delta_H^{\mathrm{i}t} H = H$$

$\mathcal{H}$ complex Hilbert space, $H \subset \mathcal{H}$ real subspace. Symplectic complement:

$$H' = \{x \in \mathcal{H} : \mathrm{Im}(x, H) = 0\} = \mathrm{i}H^{\perp}$$

Standard subspace: closed, real subspace $H \subset \mathcal{H}$ with $\overline{H + \mathrm{i}H} = \mathcal{H}$ and $H \cap \mathrm{i}H = \{0\}$.

Define antilinear unbounded closed involutive ($S^2 \subset 1$) operator

$$S_H : x + \mathrm{i}y \mapsto x - \mathrm{i}y \text{ for } x, y \in H.$$

Conversely S densely defined closed, antilinear involution on $\mathcal{H}$, $H_S = \{x \in \mathcal{H} : Sx = x\}$ is a standard subspace:

standard subspaces H	$\overset{1:1}{\longleftrightarrow}$	densely defined, closed, antilinear involutions S

Modular Theory: Polar decomposition $S_H = J_H \Delta_H^{1/2}$

$$J_H H = H' \quad \Delta_H^{\mathrm{i}t} H = H$$

$\mathcal{H}$ complex Hilbert space, $H \subset \mathcal{H}$ real subspace. Symplectic complement:

$$H' = \{x \in \mathcal{H} : \mathrm{Im}(x, H) = 0\} = \mathrm{i}H^{\perp}$$

Standard subspace: closed, real subspace $H \subset \mathcal{H}$ with $\overline{H + \mathrm{i}H} = \mathcal{H}$ and $H \cap \mathrm{i}H = \{0\}$.
Define antilinear unbounded closed involutive ($S^2 \subset 1$) operator

$$S_H : x + \mathrm{i}y \mapsto x - \mathrm{i}y \text{ for } x, y \in H.$$

Conversely S densely defined closed, antilinear involution on $\mathcal{H}$,
$H_S = \{x \in \mathcal{H} : Sx = x\}$ is a standard subspace:

standard subspaces H	$\overset{1:1}{\longleftrightarrow}$	densely defined, closed, antilinear involutions S

Modular Theory: Polar decomposition $S_H = J_H \Delta_H^{1/2}$

$$J_H H = H' \quad \Delta_H^{\mathrm{i}t} H = H$$

$\mathcal{H}$ complex Hilbert space, $H \subset \mathcal{H}$ real subspace. Symplectic complement:

$$H' = \{x \in \mathcal{H} : \mathrm{Im}(x, H) = 0\} = \mathrm{i}H^\perp$$

Standard subspace: closed, real subspace $H \subset \mathcal{H}$ with $\overline{H + \mathrm{i}H} = \mathcal{H}$ and $H \cap \mathrm{i}H = \{0\}$.
Define antilinear unbounded closed involutive $(S^2 \subset 1)$ operator

$$S_H : x + \mathrm{i}y \mapsto x - \mathrm{i}y \text{ for } x, y \in H.$$

Conversely S densely defined closed, antilinear involution on $\mathcal{H}$, $H_S = \{x \in \mathcal{H} : Sx = x\}$ is a standard subspace:

standard subspaces H	$\overset{1:1}{\longleftrightarrow}$	densely defined, closed, antilinear involutions S

Modular Theory: Polar decomposition $S_H = J_H \Delta_H^{1/2}$

$$J_H H = H' \quad \Delta_H^{\mathrm{i}t} H = H$$

$\mathcal{H}$ complex Hilbert space, $H \subset \mathcal{H}$ real subspace. Symplectic complement:

$$H' = \{x \in \mathcal{H} : \mathrm{Im}(x, H) = 0\} = \mathrm{i}H^{\perp}$$

Standard subspace: closed, real subspace $H \subset \mathcal{H}$ with $\overline{H + \mathrm{i}H} = \mathcal{H}$ and $H \cap \mathrm{i}H = \{0\}$.

Define antilinear unbounded closed involutive ($S^2 \subset 1$) operator

$$S_H : x + \mathrm{i}y \mapsto x - \mathrm{i}y \text{ for } x, y \in H.$$

Conversely S densely defined closed, antilinear involution on $\mathcal{H}$, $H_S = \{x \in \mathcal{H} : Sx = x\}$ is a standard subspace:

$$\boxed{\text{standard subspaces } H} \overset{1:1}{\longleftrightarrow} \boxed{\text{densely defined, closed, antilinear involutions } S}$$

Modular Theory: Polar decomposition $S_H = J_H \Delta_H^{1/2}$

$$J_H H = H' \quad \Delta_H^{\mathrm{i}t} H = H$$

Standard pair. (H, T)

- $H \subset \mathcal{H}$ standard subspace with
- $T(t) = e^{itP}$ one-param. group with **positive generator** P
- $T(t)H \subset H$ for $t \geq 0$

Theorem (Borchers Theorem for standard subspaces)

Let (H, T) *be a standard pair, then*

$$\Delta_H^{is} T(t) \Delta_H^{-is} = T(e^{-2\pi s} t) \qquad (s, t \in \mathbb{R})$$

$$J_H T(t) J_H = T(-t) \qquad (t \in \mathbb{R})$$

Standard pair. (H,T)

- $H \subset \mathcal{H}$ standard subspace with
- $T(t) = e^{\mathrm{i}tP}$ one-param. group with **positive generator** P
- $T(t)H \subset H$ for $t \geq 0$

Theorem (Borchers Theorem for standard subspaces)

Let (H,T) *be a standard pair, then*

$$\Delta_H^{\mathrm{i}s} T(t) \Delta_H^{-\mathrm{i}s} = T(e^{-2\pi s} t) \qquad (s,t \in \mathbb{R})$$

$$J_H T(t) J_H = T(-t) \qquad (t \in \mathbb{R})$$

$\mathcal{E}(H) =$ unitaries V on $\mathcal{H}$ such that $VH \subset H$ and $[V, T(t)] = 0$.

Analog of the Beurling-Lax theorem.

Characterization of $\mathcal{E}(H)$. (Longo and Witten, 2010)
(H, T) irreducible standard pair, then are equivalent

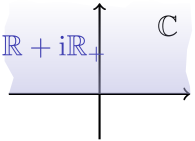
1. $V \in \mathcal{E}(H)$, i.e. $VH \subset H$ with V unitary on $\mathcal{H}$ commuting with T.
2. $V = \varphi(P)$ with φ boundary value of a symmetric inner analytic L^∞ function $\varphi : \mathbb{R} + i\mathbb{R}_+ \to \mathbb{C}$, where
   - **symmetric** $\overline{\varphi(p)} = \varphi(-p)$ for $p \geq 0$
   - **inner** $|\varphi(p)| = 1$ for $p \in \mathbb{R}$.

$\mathcal{E}(H) =$ unitaries V on $\mathcal{H}$ such that $VH \subset H$ and $[V, T(t)] = 0$.

Analog of the Beurling-Lax theorem.

Characterization of $\mathcal{E}(H)$. (Longo and Witten, 2010)
(H, T) irreducible standard pair, then are equivalent

1. $V \in \mathcal{E}(H)$, i.e. $VH \subset H$ with V unitary on $\mathcal{H}$ commuting with T.
2. $V = \varphi(P)$ with φ boundary value of a symmetric inner analytic L^∞ function $\varphi : \mathbb{R} + \mathrm{i}\mathbb{R}_+ \to \mathbb{C}$, where
   - **symmetric** $\overline{\varphi(p)} = \varphi(-p)$ for $p \geq 0$
   - **inner** $|\varphi(p)| = 1$ for $p \in \mathbb{R}$.



$\mathcal{H}$ Hilbert space, $\mathcal{I}$ = family of proper intervals on $\overline{\mathbb{R}}$

$$\mathcal{I} \ni I \longmapsto \mathcal{A}(I) = \mathcal{A}(I)'' \subset \mathrm{B}(\mathcal{H})$$

A. Isotony. $I_1 \subset I_2 \Longrightarrow \mathcal{A}(I_1) \subset \mathcal{A}(I_2)$

B. Locality. $I_1 \cap I_2 = \varnothing \Longrightarrow [\mathcal{A}(I_1), \mathcal{A}(I_2)] = \{0\}$

C. Möbius covariance. There is a unitary representation U of the Möbius group ($\cong \mathrm{PSL}(2,\mathbb{R})$) on $\mathcal{H}$ such that

$$U(g)\mathcal{A}(I)U(g)^* = \mathcal{A}(gI).$$

D. Positivity of energy. U is a positive-energy representation, i.e. generator L_0 of the rotation subgroup (conformal Hamiltonian) has positive spectrum.

E. Vacuum. $\ker L_0 = \mathbb{C}\Omega$ and Ω (vacuum vector) is a unit vector cyclic for the von Neumann algebra $\bigvee_{I\in\mathcal{I}} \mathcal{A}(I)$.

Consequences   Complete Rationality

Net of standard subspaces (prequantised theory)

- $\mathrm{L}\mathbb{R} = \mathrm{C}^\infty(S_1, \mathbb{R})$ yields a Hilbert space $\mathcal{H} = \overline{\mathrm{L}\mathbb{R}}^{\|\cdot\|}$ using
  - semi-norm. $\|f\| = \sum_{k>0} k|\hat{f}_k|$
  - complex-structure. $\mathcal{J} : \hat{f}_k \longmapsto -\mathrm{i}\,\mathrm{sign}(k)\hat{f}_k$
  - symplectic form. $\omega(f,g) = \mathrm{Im}(f,g) = 1/(4\pi)\int g df$
- Local spaces: $\mathrm{L}_I\mathbb{R} = \{f\mathrm{L}\mathbb{R} : \mathrm{supp} f \subset I\}$

$$I \longmapsto H(I) = \overline{\mathrm{L}_I\mathbb{R}} \subset \mathcal{H}$$

Conformal net of a free boson

- Second quantization. Conformal net on the symmetric Fock space $e^{\mathcal{H}}$ by CCR functor (Weyl unitaries):

$$I \longmapsto \mathcal{A}(I) := \mathrm{CCR}(H(I))'' \subset \mathrm{B}(e^{\mathcal{H}})$$

- Weyl unitaries $W(f)W(g) = e^{-\mathrm{i}\omega(f,g)}W(f+g)$,
- Vacuum state $\phi(W(f)) = (\Omega, W(f)\Omega) = e^{-1/2\|f\|^2}$

Net of standard subspaces (prequantised theory)

- $\mathrm{L}\mathbb{R} = \mathrm{C}^\infty(S_1, \mathbb{R})$ yields a Hilbert space $\mathcal{H} = \overline{\mathrm{L}\mathbb{R}}^{\|\cdot\|}$ using
  - semi-norm. $\|f\| = \sum_{k>0} k|\hat{f}_k|$
  - complex-structure. $\mathcal{J} : \hat{f}_k \longmapsto -\mathrm{i}\,\mathrm{sign}(k)\hat{f}_k$
  - symplectic form. $\omega(f, g) = \mathrm{Im}(f, g) = 1/(4\pi) \int g df$
- Local spaces: $\mathrm{L}_I\mathbb{R} = \{f\mathrm{L}\mathbb{R} : \mathrm{supp} f \subset I\}$

$$I \longmapsto H(I) = \overline{\mathrm{L}_I\mathbb{R}} \subset \mathcal{H}$$

Conformal net of a free boson

- Second quantization. Conformal net on the symmetric Fock space $e^{\mathcal{H}}$ by CCR functor (Weyl unitaries):

$$I \longmapsto \mathcal{A}(I) := \mathrm{CCR}(H(I))'' \subset \mathrm{B}(e^{\mathcal{H}})$$

- Weyl unitaries $W(f)W(g) = e^{-\mathrm{i}\omega(f,g)}W(f + g)$,
- Vacuum state $\phi(W(f)) = (\Omega, W(f)\Omega) = e^{-1/2\|f\|^2}$

Conformal net of n free bosons

$$\mathcal{A}_n(I) = \mathcal{A}_1^{\otimes n}(I) = \mathrm{CCR}(H(I) \oplus \cdots \oplus H(I))$$

Local endomorphisms (representations) of $\mathcal{A}_n = \mathcal{A}^{\otimes n}$

$\ell : S_1 \longrightarrow \mathbb{R}^n$ smooth with compact support gives **automorphism**

$$\rho_\ell(W(f)) = e^{-\frac{\mathrm{i}}{2\pi}\int \langle \ell, f\rangle_{\mathbb{R}^n}} W(f)$$

Charge:

$$q_\ell = \frac{1}{2\pi}\int_{S_1} \ell \in \mathbb{R}^n \qquad \rho_\ell \cong \rho_m \Longleftrightarrow q_\ell = q_m$$

Statistics operator:

$$\epsilon(\rho_\ell, \rho_m) = e^{\pm \mathrm{i}\pi \langle q_\ell, q_m\rangle_{\mathbb{R}^n}}$$

Local extension: If $\langle q_\ell, q_\ell\rangle \in 2\mathbb{Z}$ then $\epsilon(\rho_\ell, \rho_\ell) = 1 \rightsquigarrow$ **local extension** (by cross product).

Conformal net of n free bosons

$$\mathcal{A}_n(I) = \mathcal{A}_1^{\otimes n}(I) = \mathrm{CCR}(H(I) \oplus \cdots \oplus H(I))$$

Local endomorphisms (representations) of $\mathcal{A}_n = \mathcal{A}^{\otimes n}$

$\ell : S_1 \longrightarrow \mathbb{R}^n$ smooth with compact support gives **automorphism**

$$\rho_\ell(W(f)) = e^{-\frac{\mathrm{i}}{2\pi}\int \langle \ell, f\rangle_{\mathbb{R}^n}} W(f)$$

Charge:

$$q_\ell = \frac{1}{2\pi}\int_{S_1} \ell \in \mathbb{R}^n \qquad \rho_\ell \cong \rho_m \Longleftrightarrow q_\ell = q_m$$

Statistics operator:

$$\epsilon(\rho_\ell, \rho_m) = e^{\pm \mathrm{i}\pi \langle q_\ell, q_m\rangle_{\mathbb{R}^n}}$$

Local extension: If $\langle q_\ell, q_\ell\rangle \in 2\mathbb{Z}$ then $\epsilon(\rho_\ell, \rho_\ell) = 1 \rightsquigarrow$ **local extension** (by cross product).

Conformal net of n free bosons

$$\mathcal{A}_n(I) = \mathcal{A}_1^{\otimes n}(I) = \mathrm{CCR}(H(I) \oplus \cdots \oplus H(I))$$

Local endomorphisms (representations) of $\mathcal{A}_n = \mathcal{A}^{\otimes n}$
$\ell : S_1 \longrightarrow \mathbb{R}^n$ smooth with compact support gives **automorphism**

$$\rho_\ell(W(f)) = e^{-\frac{\mathrm{i}}{2\pi}\int \langle \ell, f \rangle_{\mathbb{R}^n}} W(f)$$

Charge:

$$q_\ell = \frac{1}{2\pi}\int_{S_1} \ell \in \mathbb{R}^n \qquad \rho_\ell \cong \rho_m \Longleftrightarrow q_\ell = q_m$$

Statistics operator:

$$\epsilon(\rho_\ell, \rho_m) = e^{\pm \mathrm{i}\pi \langle q_\ell, q_m \rangle_{\mathbb{R}^n}}$$

Local extension: If $\langle q_\ell, q_\ell \rangle \in 2\mathbb{Z}$ then $\epsilon(\rho_\ell, \rho_\ell) = 1 \rightsquigarrow$ **local extension** (by cross product).

Conformal net of n free bosons

$$\mathcal{A}_n(I) = \mathcal{A}_1^{\otimes n}(I) = \mathrm{CCR}(H(I) \oplus \cdots \oplus H(I))$$

Local endomorphisms (representations) of $\mathcal{A}_n = \mathcal{A}^{\otimes n}$
$\ell : S_1 \longrightarrow \mathbb{R}^n$ smooth with compact support gives **automorphism**

$$\rho_\ell(W(f)) = e^{-\frac{\mathrm{i}}{2\pi}\int \langle \ell, f\rangle_{\mathbb{R}^n}} W(f)$$

Charge:

$$q_\ell = \frac{1}{2\pi}\int_{S_1} \ell \in \mathbb{R}^n \qquad \rho_\ell \cong \rho_m \Longleftrightarrow q_\ell = q_m$$

Statistics operator:

$$\epsilon(\rho_\ell, \rho_m) = e^{\pm \mathrm{i}\pi \langle q_\ell, q_m\rangle_{\mathbb{R}^n}}$$

Local extension: If $\langle q_\ell, q_\ell\rangle \in 2\mathbb{Z}$ then $\epsilon(\rho_\ell, \rho_\ell) = 1 \rightsquigarrow$ **local extension** (by cross product).

Conformal net of n free bosons

$$\mathcal{A}_n(I) = \mathcal{A}_1^{\otimes n}(I) = \mathrm{CCR}(H(I) \oplus \cdots \oplus H(I))$$

Local endomorphisms (representations) of $\mathcal{A}_n = \mathcal{A}^{\otimes n}$
$\ell : S_1 \longrightarrow \mathbb{R}^n$ smooth with compact support gives **automorphism**

$$\rho_\ell(W(f)) = e^{-\frac{\mathrm{i}}{2\pi}\int \langle \ell, f\rangle_{\mathbb{R}^n}} W(f)$$

Charge:

$$q_\ell = \frac{1}{2\pi}\int_{S_1} \ell \in \mathbb{R}^n \qquad \rho_\ell \cong \rho_m \Longleftrightarrow q_\ell = q_m$$

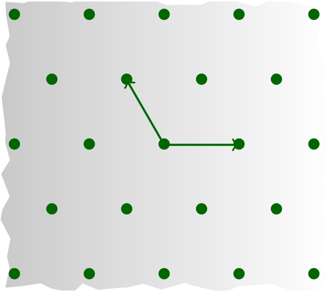
Statistics operator:

$$\epsilon(\rho_\ell, \rho_m) = e^{\pm \mathrm{i}\pi \langle q_\ell, q_m\rangle_{\mathbb{R}^n}}$$

Local extension: If $\langle q_\ell, q_\ell\rangle \in 2\mathbb{Z}$ then $\epsilon(\rho_\ell, \rho_\ell) = 1 \rightsquigarrow$ **local extension** (by cross product).

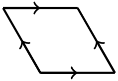
Let Q be an (positive-definite) **even lattice** (eg. root lattice) of rank n

- $\forall \alpha \in Q$: $\langle \alpha, \alpha \rangle \in 2\mathbb{N} \Longrightarrow$ integral $\forall \alpha, \beta \in Q$: $\langle \alpha, \beta \rangle \in \mathbb{Z}$.
- **dual lattice** (characters) $Q^* = \{\alpha \in \mathbb{R}^n : \langle \alpha, Q \rangle \in \mathbb{Z}\}$ (eg. weight lattice in case of root lattices).



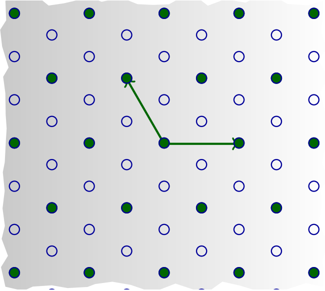
$\mathrm{A}_2 \leftrightarrow \mathrm{SU}(3)$

corresponding **torus** $T_Q = (Q \otimes_{\mathbb{Z}} \mathbb{C})/Q$



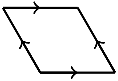
Let Q be an (positive-definite) **even lattice** (eg. root lattice) of rank n

- $\forall \alpha \in Q$: $\langle \alpha, \alpha \rangle \in 2\mathbb{N} \Longrightarrow$ integral $\forall \alpha, \beta \in Q$: $\langle \alpha, \beta \rangle \in \mathbb{Z}$.
- **dual lattice** (characters) $Q^* = \{\alpha \in \mathbb{R}^n : \langle \alpha, Q \rangle \in \mathbb{Z}\}$ (eg. weight lattice in case of root lattices).



$\mathrm{A}_2 \leftrightarrow \mathrm{SU}(3)$

corresponding **torus**
$T_Q = (Q \otimes_{\mathbb{Z}} \mathbb{C})/Q$



Local extension. For a lattice Q of rank n there is $\mathcal{A}_Q \supset \mathcal{A}^{\otimes n}$ containing of the net $\equiv \mathcal{A}^{\otimes n}$ of n free bosons. Locally

$$\mathcal{A}_Q(I) = (\mathcal{A}(I) \otimes \ldots \otimes \mathcal{A}(I)) \rtimes Q$$

(Buchholz, Mack, Todorov 1988) $(n = 1)$ (Staszkiewicz, 1995) (Dong and Xu, 2006)

Construction

even lattice Q of rank n	$\longrightarrow$	Completely rational conformal net $\mathcal{A}_Q$

- Conformal nets corresponding to Lattice Vertex Operator Algebras.

Some properties:

- Sectors finite group Q^*/Q, each sector statistical dimension 1.
- Completely rational net $\mu = |Q^*/Q|$ (Dong and Xu, 2006).

Local extension. For a lattice Q of rank n there is $\mathcal{A}_Q \supset \mathcal{A}^{\otimes n}$ containing of the net $\equiv \mathcal{A}^{\otimes n}$ of n free bosons. Locally

$$\mathcal{A}_Q(I) = (\mathcal{A}(I) \otimes \ldots \otimes \mathcal{A}(I)) \rtimes Q$$

(Buchholz, Mack, Todorov 1988) $(n = 1)$ (Staszkiewicz, 1995) (Dong and Xu, 2006)

Construction

even lattice Q of rank n	$\longrightarrow$	Completely rational conformal net $\mathcal{A}_Q$

- Conformal nets corresponding to Lattice Vertex Operator Algebras.

Some properties:

- Sectors finite group Q^*/Q, each sector statistical dimension 1.
- Completely rational net $\mu = |Q^*/Q|$ (Dong and Xu, 2006).

Local extension. For a lattice Q of rank n there is $\mathcal{A}_Q \supset \mathcal{A}^{\otimes n}$ containing of the net $\equiv \mathcal{A}^{\otimes n}$ of n free bosons. Locally

$$\mathcal{A}_Q(I) = (\mathcal{A}(I) \otimes \ldots \otimes \mathcal{A}(I)) \rtimes Q$$

(Buchholz, Mack, Todorov 1988) $(n = 1)$ (Staszkiewicz, 1995) (Dong and Xu, 2006)

Construction

even lattice Q of rank n	$\longrightarrow$	Completely rational conformal net $\mathcal{A}_Q$

- Conformal nets corresponding to Lattice Vertex Operator Algebras.

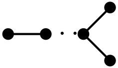
Some properties:

- Sectors finite group Q^*/Q, each sector statistical dimension 1.
- Completely rational net $\mu = |Q^*/Q|$ (Dong and Xu, 2006).

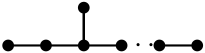
G simply-connected **simple-laced** Lie group, e.g.

A $\mathrm{SU}(n+1)$, $n \geq 1 \leftrightarrow \mathrm{A}_n$:

D $\mathrm{Spin}(2n)$, $n \geq 3 \leftrightarrow \mathrm{D}_n$:



E Exceptional Lie Groups $\mathrm{E}_6, \mathrm{E}_7, \mathrm{E}_8$:



Q **root lattice** spanned by simple roots $\{\alpha_1, \ldots, \alpha_n\}$

Cartan matrix (C_{ij}) $\quad \langle \alpha_i, \alpha_j \rangle = C_{ij} = \begin{cases} 2 & i = j \\ -1 & i \bullet\!\!-\!\!\bullet j \\ 0 & \end{cases}$

G simply-connected **simple-laced** Lie group, e.g.

- A $\mathrm{SU}(n+1)$, $n \geq 1 \leftrightarrow \mathrm{A}_n$
- D $\mathrm{Spin}(2n)$, $n \geq 3 \leftrightarrow \mathrm{D}_n$
- E Exceptional Lie Groups $\mathrm{E}_6, \mathrm{E}_7, \mathrm{E}_8$

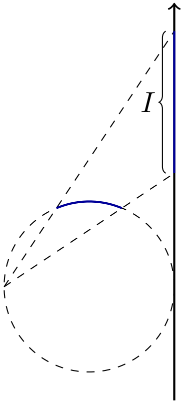
Q **root lattice** spanned by simple roots $\{\alpha_1, \ldots, \alpha_n\}$
Maximal torus $(Q \otimes_{\mathbb{Z}} \mathbb{R})/Q \cong T \subset G \rightsquigarrow \mathcal{A}_{T,1} \equiv \mathcal{A}_Q$
(Conjectured) equivalence (proofed in case $G = \mathrm{SU}(n)$ (Xu, 2009))

$$\boxed{\begin{array}{c}\text{loop group net}\\ \text{for such } G \text{ at level 1}\end{array}} = \mathcal{A}_{G,1} \overset{\sim}{\longleftrightarrow} \mathcal{A}_Q = \boxed{\begin{array}{c}\text{conformal net}\\ \text{associated at } Q\end{array}}$$

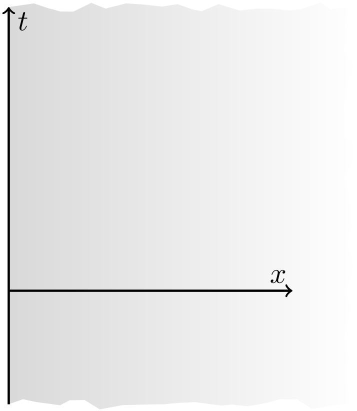
Outline

- Conformal net on the **real line** identifying $S_1 \setminus \{-1\} \cong \mathbb{R}$

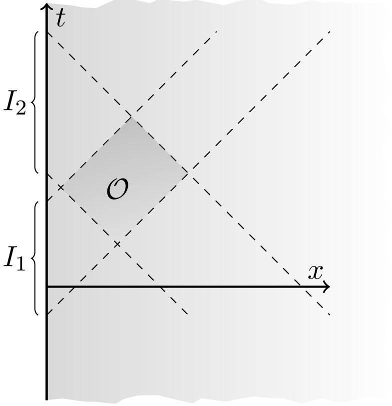
Conformal net on S_1	$\xrightarrow{\text{restriction}}$	Conformal net on $\mathbb{R}$

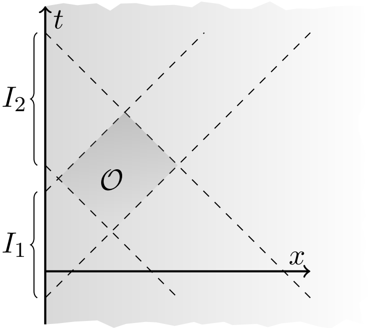


- **Minkowski half-plane** $x > 0$, $ds^2 = dt^2 - dx^2$
- **Double cone** $\mathcal{O} = I_1 \times I_2$ where I_1, I_2 disjoint intervals



- **Minkowski half-plane** $x > 0$, $ds^2 = dt^2 - dx^2$
- **Double cone** $\mathcal{O} = I_1 \times I_2$ where I_1, I_2 disjoint intervals





Boundary conformal quantum field theory (Longo and Rehren, 2004)

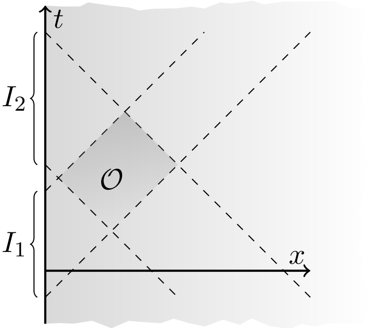
$$\mathcal{A}_+(\mathcal{O}) = \mathcal{A}(I_1) \vee \mathcal{A}(I_2)$$

Boundary quantum field theory (Longo and Witten, 2010)

$$\mathcal{A}_V(\mathcal{O}) = \mathcal{A}(I_1) \vee V\mathcal{A}(I_2)V^*$$

V unitary on $\mathcal{H}$

- $[V, T(t)] = 0$, i.e. commutes with translation $T(t)$
- $V\mathcal{A}(\mathbb{R}_+)V^* \subset \mathcal{A}(\mathbb{R}_+)$



Boundary conformal quantum field theory (Longo and Rehren, 2004)

$$\mathcal{A}_+(\mathcal{O}) = \mathcal{A}(I_1) \vee \mathcal{A}(I_2)$$

Boundary quantum field theory (Longo and Witten, 2010)

$$\mathcal{A}_V(\mathcal{O}) = \mathcal{A}(I_1) \vee V\mathcal{A}(I_2)V^*$$

V unitary on $\mathcal{H}$

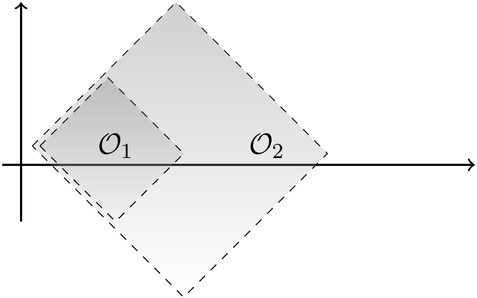
- $[V, T(t)] = 0$, i.e. commutes with translation $T(t)$
- $V\mathcal{A}(\mathbb{R}_+)V^* \subset \mathcal{A}(\mathbb{R}_+)$

A local (time) translation covariant net on Minkowski half-plane on a Hilbert space $\mathcal{H}$ is a map $\mathcal{K}_+ \ni \mathcal{O} \longmapsto \mathcal{B}(\mathcal{O}) \subset \mathrm{B}(\mathcal{H})$ which fulfills:

1. Isotony. $\mathcal{O}_1 \subset \mathcal{O}_2$ implies $\mathcal{B}(\mathcal{O}_1) \subset \mathcal{B}(\mathcal{O}_2)$.
2. Locality. If $\mathcal{O}_1, \mathcal{O}_2 \in \mathcal{K}_+$ are mutually space-like separated then $[\mathcal{B}(\mathcal{O}_1), \mathcal{B}(\mathcal{O}_2)] = \{0\}$.

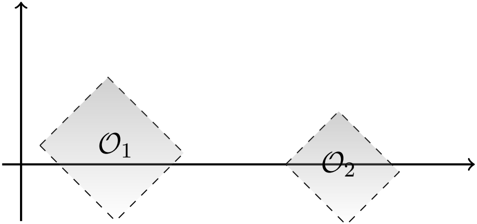
A local (time) translation covariant net on Minkowski half-plane on a Hilbert space $\mathcal{H}$ is a map $\mathcal{K}_+ \ni \mathcal{O} \longmapsto \mathcal{B}(\mathcal{O}) \subset \mathrm{B}(\mathcal{H})$ which fulfills:

1. **Isotony.** $\mathcal{O}_1 \subset \mathcal{O}_2$ implies $\mathcal{B}(\mathcal{O}_1) \subset \mathcal{B}(\mathcal{O}_2)$.
2. **Locality.** If $\mathcal{O}_1, \mathcal{O}_2 \in \mathcal{K}_+$ are mutually space-like separated then $[\mathcal{B}(\mathcal{O}_1), \mathcal{B}(\mathcal{O}_2)] = \{0\}$.



A local (time) translation covariant net on Minkowski half-plane on a Hilbert space $\mathcal{H}$ is a map $\mathcal{K}_+ \ni \mathcal{O} \longmapsto \mathcal{B}(\mathcal{O}) \subset \mathrm{B}(\mathcal{H})$ which fulfills:

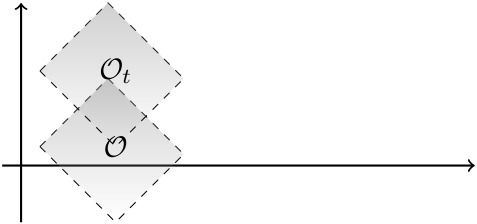
1. Isotony. $\mathcal{O}_1 \subset \mathcal{O}_2$ implies $\mathcal{B}(\mathcal{O}_1) \subset \mathcal{B}(\mathcal{O}_2)$.
2. Locality. If $\mathcal{O}_1, \mathcal{O}_2 \in \mathcal{K}_+$ are mutually space-like separated then $[\mathcal{B}(\mathcal{O}_1), \mathcal{B}(\mathcal{O}_2)] = \{0\}$.



A local (time) translation covariant net on Minkowski half-plane on a Hilbert space $\mathcal{H}$ is a map $\mathcal{K}_+ \ni \mathcal{O} \longmapsto \mathcal{B}(\mathcal{O}) \subset \mathrm{B}(\mathcal{H})$ which fulfills:

1. **Isotony.** $\mathcal{O}_1 \subset \mathcal{O}_2$ implies $\mathcal{B}(\mathcal{O}_1) \subset \mathcal{B}(\mathcal{O}_2)$.
2. **Locality.** If $\mathcal{O}_1, \mathcal{O}_2 \in \mathcal{K}_+$ are mutually space-like separated then $[\mathcal{B}(\mathcal{O}_1), \mathcal{B}(\mathcal{O}_2)] = \{0\}$.
3. **Time-translation covariance** $\exists$ an unitary one-parameter group $T(t) = e^{\mathrm{i}tP}$ with positive generator P such that:

$$T(t)\mathcal{B}(\mathcal{O})T(t)^* = \mathcal{B}(\mathcal{O}_t), \qquad \mathcal{O} \in \mathcal{K}_+, \qquad \mathcal{O}_t = \mathcal{O} + (t, 0)$$



A local (time) translation covariant net on Minkowski half-plane on a Hilbert space $\mathcal{H}$ is a map $\mathcal{K}_+ \ni \mathcal{O} \longmapsto \mathcal{B}(\mathcal{O}) \subset \mathrm{B}(\mathcal{H})$ which fulfills:

1. **Isotony.** $\mathcal{O}_1 \subset \mathcal{O}_2$ implies $\mathcal{B}(\mathcal{O}_1) \subset \mathcal{B}(\mathcal{O}_2)$.
2. **Locality.** If $\mathcal{O}_1, \mathcal{O}_2 \in \mathcal{K}_+$ are mutually space-like separated then $[\mathcal{B}(\mathcal{O}_1), \mathcal{B}(\mathcal{O}_2)] = \{0\}$.
3. **Time-translation covariance** $\exists$ an unitary one-parameter group $T(t) = e^{\mathrm{i}tP}$ with positive generator P such that:

$$T(t)\mathcal{B}(\mathcal{O})T(t)^* = \mathcal{B}(\mathcal{O}_t), \qquad \mathcal{O} \in \mathcal{K}_+, \qquad \mathcal{O}_t = \mathcal{O} + (t, 0)$$

4. **Vacuum.** $\Omega \in \mathcal{H}$ is a up to the multiple unique T invariant vector and cyclic and separating for every $\mathcal{B}(\mathcal{O})$ for $\mathcal{O} \in \mathcal{K}_+$.

Semigroup $\mathcal{E}(\mathcal{A})$ associated to a conformal net $\mathcal{A}$

Semigroup $\mathcal{E}(\mathcal{A})$ of unitaries on $\mathcal{H}$ (associated to $\mathcal{A}$)

- $[V, T(t)] = 0$, i.e. commutes with translation $T(t)$
- $V\mathcal{A}(\mathbb{R}_+)V^* \subset \mathcal{A}(\mathbb{R}_+) \rightsquigarrow V\mathcal{A}(a + \mathbb{R}_+)V^* \subset \mathcal{A}(\mathbb{R}_+)$

Trivial examples of elements in $\mathcal{E}(\mathcal{A})$:

- $V = T(t)$ $t > 0$ positive **translations**
- V **inner symmetry**, i.e $V\mathcal{A}(I)V^* = \mathcal{A}(I)$ for all proper I

Construction

Conformal net $\mathcal{A}$ on $\mathbb{R}$	$+$	semigroup element $V \in \mathcal{E}(\mathcal{A})$	$\longrightarrow$	local net $\mathcal{A}_V$ on M_+

$\mathcal{H}$ one-particle space of a bosons (completion of $\mathrm{L}\mathbb{R}$) $H(\mathbb{R}_+)$ standard subspace localized in $\mathbb{R}_+$
$\varphi : \mathbb{R} \longrightarrow \mathbb{C}$ inner function, then

$$V_0 = \varphi(P_0) \Longrightarrow V_0 H(\mathbb{R}_+) \subset H(\mathbb{R}_+),\ [V_0, e^{\mathrm{i}tP_0}] = 0$$

P_0 generator of translation.
By **second quantization** $\mathcal{A}(I) = \mathrm{CCR}(H(I))''$.

$$V = \Gamma(V_0) \Longrightarrow V \in \mathcal{E}(\mathcal{A})$$

More general for n bosons

$$\mathcal{A}_n(\mathbb{R}_+) \cong \mathcal{A}(\mathbb{R}_+)^{\otimes n} = \mathrm{CCR}(H(\mathbb{R}_+) \oplus \cdots \oplus H(\mathbb{R}_+))''$$

Theorem (Prequantized semigroup reducible case (Longo and Witten, 2010))

$V_0 \in \mathcal{E}(H(\mathbb{R}_+) \oplus \cdots \oplus H(\mathbb{R}_+))$, then $V_0 = \varphi_{kl}(P_0)$ matrices of functions such that $\varphi_{kl}(p)$ unitary matrix for almost all $p > 0$, φ_{kl} boundary value of a L^∞ function analytic on the upper half-plane which is symmetric $\overline{\varphi_{kl}(p)} = \varphi_{kl}(-p)$.

Theorem

$V = \Gamma(V_0) \in \mathcal{E}(\mathcal{A}_n)$ for the second quantization of V_0 given above.

More general for n bosons

$$\mathcal{A}_n(\mathbb{R}_+) \cong \mathcal{A}(\mathbb{R}_+)^{\otimes n} = \mathrm{CCR}(H(\mathbb{R}_+) \oplus \cdots \oplus H(\mathbb{R}_+))''$$

Theorem (Prequantized semigroup reducible case (Longo and Witten, 2010))

$V_0 \in \mathcal{E}(H(\mathbb{R}_+) \oplus \cdots \oplus H(\mathbb{R}_+))$, then $V_0 = \varphi_{kl}(P_0)$ matrices of functions such that $\varphi_{kl}(p)$ unitary matrix for almost all $p > 0$, φ_{kl} boundary value of a L^∞ function analytic on the upper half-plane which is symmetric $\overline{\varphi_{kl}(p)} = \varphi_{kl}(-p)$.

Theorem

$V = \Gamma(V_0) \in \mathcal{E}(\mathcal{A}_n)$ for the second quantization of V_0 given above.

Question

Which elements of the semigroup $\mathcal{E}(\mathcal{A}_n)$ extend to the local extensions by lattices?

$$\mathcal{A}_Q(I) = \mathcal{A}_n(I) \rtimes Q$$

where Q even lattice of rank n

Extension of the endomorphism $\eta = \mathrm{Ad} V$ of $\mathcal{A}_n(\mathbb{R}_+)$ with $V \in \mathcal{E}(\mathcal{A}_n)$ to

$$\mathcal{A}_Q(\mathbb{R}_+) = \mathcal{A}_n(\mathbb{R}_+) \rtimes_{\beta_i} Q$$

β_i localized in $\mathbb{R}_+$

Assume η and β_i **commute up to some cocycle** $z_i \in \mathcal{A}_n(\mathbb{R}_+)$

$$z_i \in \mathrm{Hom}(\eta\beta_i, \beta_i\eta) \Longleftrightarrow z_i\beta_i(\eta(x)) = \eta(\beta_i(x))z_i \quad \text{for all } x \in \mathcal{A}_n(\mathbb{R}_+)$$

and the **compatibility condition**

$$z_i\beta_i(z_j) = z_j\beta_j(z_i)$$

then η extends to $\tilde{\eta} = \mathrm{Ad}\tilde{V}$.

$$V \in \mathcal{E}(\mathcal{A}_n) \xrightarrow{\text{extends}} \tilde{V} \in \mathcal{E}(\mathcal{A}_Q)$$

Extension of the endomorphism $\eta = \mathrm{Ad} V$ of $\mathcal{A}_n(\mathbb{R}_+)$ with $V \in \mathcal{E}(\mathcal{A}_n)$ to

$$\mathcal{A}_Q(\mathbb{R}_+) = \mathcal{A}_n(\mathbb{R}_+) \rtimes_{\beta_i} Q$$

β_i localized in $\mathbb{R}_+$
Assume η and β_i **commute up to some cocycle** $z_i \in \mathcal{A}_n(\mathbb{R}_+)$

$$z_i \in \mathrm{Hom}(\eta\beta_i, \beta_i\eta) \iff z_i\beta_i(\eta(x)) = \eta(\beta_i(x))z_i \quad \text{for all } x \in \mathcal{A}_n(\mathbb{R}_+)$$

and the **compatibility condition**

$$z_i\beta_i(z_j) = z_j\beta_j(z_i)$$

then η extends to $\tilde{\eta} = \mathrm{Ad}\tilde{V}$.

$$V \in \mathcal{E}(\mathcal{A}_n) \xrightarrow{\text{extends}} \tilde{V} \in \mathcal{E}(\mathcal{A}_Q)$$

Existence of $z_i \in \mathrm{Hom}_{\mathcal{A}_n(\mathbb{R}_+)}(\eta\beta_i, \beta_i\eta)$ with the above properties in our model ensure

$$V = \Gamma(\varphi_{ik}(P_0)) \in \mathcal{E}(\mathcal{A}_n) \xrightarrow{\text{extends?}} \tilde{V} \in \mathcal{E}(\mathcal{A}_Q)$$

Restrictions. Such z_i can be constructed if

- Algebraic obstruction. The "inner function matrix" has to be constant on every component of the lattice
- Analytical obstruction. The "inner function" need to be Hölder continuous at 0, i.e.

$$\frac{|1 - \varphi(p)|^2}{|p|} \text{ locally integrable at } p = 0$$

Existence of $z_i \in \mathrm{Hom}_{\mathcal{A}_n(\mathbb{R}_+)}(\eta\beta_i, \beta_i\eta)$ with the above properties in our model ensure

$$V = \Gamma(\varphi_{ik}(P_0)) \in \mathcal{E}(\mathcal{A}_n) \xrightarrow{\text{extends?}} \tilde{V} \in \mathcal{E}(\mathcal{A}_Q)$$

Restrictions. Such z_i can be constructed if

- Algebraic obstruction. The “inner function matrix” has to be constant on every component of the lattice
- Analytical obstruction. The “inner function” need to be Hölder continuous at 0, i.e.

$$\frac{|1-\varphi(p)|^2}{|p|} \text{ locally integrable at } p = 0$$

Existence of $z_i \in \mathrm{Hom}_{\mathcal{A}_n(\mathbb{R}_+)}(\eta\beta_i, \beta_i\eta)$ with the above properties in our model ensure

$$V = \Gamma(\varphi_{ik}(P_0)) \in \mathcal{E}(\mathcal{A}_n) \xrightarrow{\text{extends?}} \tilde{V} \in \mathcal{E}(\mathcal{A}_Q)$$

Restrictions. Such z_i can be constructed if

- Algebraic obstruction. The "inner function matrix" has to be constant on every component of the lattice
- Analytical obstruction. The "inner function" need to be Hölder continuous at 0, i.e.

$$\frac{|1-\varphi(p)|^2}{|p|} \text{ locally integrable at } p = 0$$

Theorem

Let $\mathcal{A}$ be conformal net of the family

- $\mathcal{A}_Q$ *associated to an even irreducible lattice* Q
- $\mathcal{A}_{G,1}$ *for* $G = \mathrm{SU}(n)$ *(*G *simple, simply connected, simple-laced)*

$\mathcal{A}$ and φ Hölder cont.	$\longrightarrow$	$V \in \mathcal{E}(\mathcal{A})$	$\longrightarrow$	*local net $\mathcal{A}_V$ on Minkowski half-plane*

Further

- U inner symmetry $V \in \mathcal{E}(\mathcal{A}) \Longrightarrow VU \in \mathcal{E}(\mathcal{A})$
- $V_i \in \mathcal{E}(\mathcal{A}_i) \Longrightarrow V_1 \otimes \ldots \otimes V_n \in \mathcal{E}(\mathcal{A}_1 \otimes \cdots \otimes \mathcal{A}_n)$

Theorem

Let $\mathcal{A}$ be conformal net of the family

- $\mathcal{A}_Q$ *associated to an even irreducible lattice* Q
- $\mathcal{A}_{G,1}$ *for* $G = \mathrm{SU}(n)$ *(*G *simple, simply connected, simple-laced)*

$\mathcal{A}$ *and* φ *Hölder cont.*	$\longrightarrow$	$V \in \mathcal{E}(\mathcal{A})$	$\longrightarrow$	*local net* $\mathcal{A}_V$ *on Minkowski half-plane*

Further

- U inner symmetry $V \in \mathcal{E}(\mathcal{A}) \Longrightarrow VU \in \mathcal{E}(\mathcal{A})$
- $V_i \in \mathcal{E}(\mathcal{A}_i) \Longrightarrow V_1 \otimes \ldots \otimes V_n \in \mathcal{E}(\mathcal{A}_1 \otimes \cdots \otimes \mathcal{A}_n)$

We have constructed

- Elements of the semigroup $\mathcal{E}(\mathcal{A})$ for a large class of rational conformal field theories is found

➔ New models of boundary quantum field theory.

Open questions

- Loop group nets at higher level (Coset construction/Orbifold)
- Restriction of a net of free fermions (semigroup elements by second quantization) should give more examples.
- Construction of 1+1D massive models one-parameter semigroup. Until yet just examples from free field construction.

Thank you!!

Buchholz, D., Mack, G., and Todorov, I. (1988). The current algebra on the circle as a germ of local field theories. *Nuclear Physics B-Proceedings Supplements*, 5(2):20–56.

Dong, C. and Xu, F. (2006). Conformal nets associated with lattices and their orbifolds. *Adv. Math.*, 206(1):279–306.

Kawahigashi, Y. and Longo, R. (2004). Classification of local conformal nets. Case c¡ 1. *The Annals of Mathematics*, 160(2):493–522.

Kawahigashi, Y., Longo, R., and Müger, M. (2001). Multi-Interval Subfactors and Modularityof Representations in Conformal Field Theory. *Communications in Mathematical Physics*, 219:631–669.

Longo, R. and Rehren, K. (2004). Local Fields in Boundary Conformal QFT. *Reviews in Mathematical Physics*, 16:909–960.

Longo, R. and Witten, E. (2010). An Algebraic Construction of Boundary Quantum Field Theory. *Communications in Mathematical Physics*, 1:179.

Staszkiewicz, C. (1995). *Die lokale Struktur abelscher Stromalgebren auf dem Kreis*. PhD thesis, Freie Universität Berlin.

Xu, F. (2000). Jones-wassermann subfactors for disconnected intervals. *Communications in Contemporary Mathematics*, 2(3):307–347.

Xu, F. (2009). On affine orbifold nets associated with outer automorphisms. *Communications in Mathematical Physics*, 291:845–861. 10.1007/s00220-009-0763-y.

- Irreducibility. $\bigvee_{I\in\mathcal{I}} \mathcal{A}(I) = B(H)$
- Reeh-Schlieder theorem. Ω is cyclic and separating for each $\mathcal{A}(I)$.
- Bisognano-Wichmann property. The Tomita-Takesaki modular operator Δ_I and and conjugation J_I of the pair $(\mathcal{A}(I), \Omega)$ are

$$U(\Lambda(-2\pi t)) = \Delta^{\mathrm{it}},\ t \in \mathbb{R} \qquad \text{dilation}$$

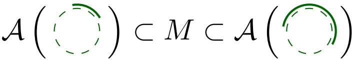
$$U(r_I) = J_I \qquad \text{reflection}$$

  (Frölich-Gabbiani, Guido-Longo)
- Haag duality. $\mathcal{A}(I') = A(I)'$.
- Factoriality. $\mathcal{A}(I)$ is III_1-factor (in Connes classification)
- Additivity. $I \subset \bigcup_i U_i \Longrightarrow A(I) \subset \bigvee_i A(I_i)$ (Fredenhagen, Jorss).

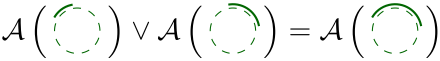
◂ Back

Completely rational conformal net (Kawahigashi, Longo, Müger 2001)

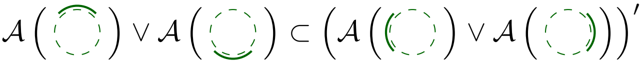
- Split property. For every relatively compact inclusion of intervalls $\exists$ intermediate **type** I **factor** M



- Strong additivity. Additivity for touching intervals:



- Finite μ-index: finite Jones index of subfactor



where the intervals are splitting the circle.

- Only finite sectors with finite statistical dimension
- Modularity: The category of DHR sectors is modular, i.e. non degenerated braiding.

Back

G compact Lie group

Loop group: $\mathrm{L}G = \mathrm{C}^\infty(S_1, G)$ (point wise multiplication)

Projective representations $\longleftrightarrow$ representations of a central extension

$$1 \longrightarrow \mathbb{T} \longrightarrow \widetilde{\mathrm{L}G} \longrightarrow \mathrm{L}G \longrightarrow 1$$

$\pi_{0,k}$ projective **positive-energy** and **vacuum** representation (classified by the level k)

$$I \longmapsto \mathcal{A}_{G,k}(I) = \pi_{0,k}(\mathrm{L}_I G)''$$

is a **conformal net**; $\mathrm{L}_I G$ loops supported in I.

E.g. $G = \mathrm{SU}(n)$ gives completely rational conformal net (Xu, 2000)

G compact Lie group
Loop group: $\mathrm{L}G = \mathrm{C}^\infty(S_1, G)$ (point wise multiplication)
Projective representations $\longleftrightarrow$ representations of a central extension

$$1 \longrightarrow \mathbb{T} \longrightarrow \widetilde{\mathrm{L}G} \longrightarrow \mathrm{L}G \longrightarrow 1$$

$\pi_{0,k}$ projective **positive-energy** and **vacuum** representation (classified by the level k)

$$I \longmapsto \mathcal{A}_{G,k}(I) = \pi_{0,k}(\mathrm{L}_I G)''$$

is a **conformal net**; $\mathrm{L}_I G$ loops supported in I.
E.g. $G = \mathrm{SU}(n)$ gives completely rational conformal net (Xu, 2000)

G compact Lie group
Loop group: $\mathrm{L}G = \mathrm{C}^\infty(S_1, G)$ (point wise multiplication)
Projective representations $\longleftrightarrow$ representations of a central extension

$$1 \longrightarrow \mathbb{T} \longrightarrow \widetilde{\mathrm{L}G} \longrightarrow \mathrm{L}G \longrightarrow 1$$

$\pi_{0,k}$ projective **positive-energy** and **vacuum** representation (classified by the level k)

$$I \longmapsto \mathcal{A}_{G,k}(I) = \pi_{0,k}(\mathrm{L}_I G)''$$

is a **conformal net**; $\mathrm{L}_I G$ loops supported in I.
E.g. $G = \mathrm{SU}(n)$ gives completely rational conformal net (Xu, 2000)

G compact Lie group
Loop group: $\mathrm{L}G = \mathrm{C}^\infty(S_1, G)$ (point wise multiplication)
Projective representations $\longleftrightarrow$ representations of a central extension

$$1 \longrightarrow \mathbb{T} \longrightarrow \widetilde{\mathrm{L}G} \longrightarrow \mathrm{L}G \longrightarrow 1$$

$\pi_{0,k}$ projective **positive-energy** and **vacuum** representation (classified by the level k)

$$I \longmapsto \mathcal{A}_{G,k}(I) = \pi_{0,k}(\mathrm{L}_I G)''$$

is a **conformal net**; $\mathrm{L}_I G$ loops supported in I.
E.g. $G = \mathrm{SU}(n)$ gives completely rational conformal net (Xu, 2000)