Generalized fixed points of conformal nets *

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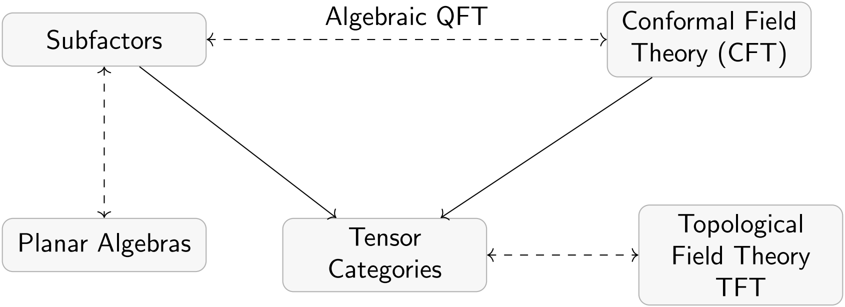

In binding together elements long-known but heretofore scattered and appearing unrelated to one another, it suddenly brings order where there reigned apparent chaos — Henri Poincaré

Question [Evans–Gannon '11]

Can we orbifold[a] a VOA [or conformal net] by something more general than a group?

[a] **orbifold** = fixed point by a finite group of (gauge) automorphisms

- Completely positive maps naturally generalize gauge transformations.
- Hypergroups of completely positive maps are generalized symmetries of quantum field theory in low dimensions.
- Finite index subtheories can be described as fixed points by hypergroup actions.

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ϕ **extremal** if $\phi = t\phi_1 + (1-t)\phi_2$ for $t \in (0,1)$, ϕ_i Markov $\Rightarrow \phi_i = \phi$.

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- Fusion algebras with $c_k = \frac{[\rho_k]}{\mathrm{FPdim}(\rho_k)}$, e.g. $K_0(\mathcal{F})$ for $\mathcal{F}$ fusion category.

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Example

Finite group G of outer gauge automorphisms $\{\alpha_g : g \in G\}$ with Haar element

$$E(\,\cdot\,) = \frac{1}{|G|} \sum_{g \in G} \alpha_g$$

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Then there exists a unique finite hypergroup K of Markov maps, such that $N = M^K$. The global weight of K is $D(K) \equiv \sum_i w_i = [M : N]$.

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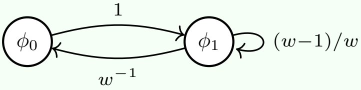

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In this case, $P = M^L$ and $N = P^{K/\!/L}$

$$\begin{array}{ccccc} M^K & \subset & M^L & \subset & M \\ \| & & \| & & \| \\ N & \subset & P & \subset & M \\ \| & & \| & & \\ P^{K/\!/L} & \subset & P & & \end{array}$$

where hypergroup $K/\!/L$ acts properly on P

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due to Jones' index rigidity $[M : N] \in \{4\cos^2(\pi/n) : n = 3, 4, 5, \ldots\} \cup [4, \infty]$ and $[M : N] \leq 5$ classification. New gap $(3 + \sqrt{3}, 5)$ cf. [Carpi–Kawahigashi–Longo '10])

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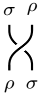
$\mathcal{C}$ UFC with **braiding** $\{\varepsilon(\rho,\sigma) \colon \rho \otimes \sigma \to \sigma \otimes \rho\}$ denoted $\varepsilon_{\rho,\sigma} =$ 

Müger's centralizer $\mathcal{D} \subset \mathcal{C}$ subcategory of $\mathcal{C}$ braided fusion category:

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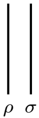
$$\stackrel{!}{=}$$ 

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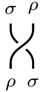
$\mathcal{C}$ UFC with **braiding** $\{\varepsilon(\rho, \sigma) \colon \rho \otimes \sigma \to \sigma \otimes \rho\}$ denoted $\varepsilon_{\rho,\sigma} =$ 

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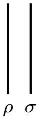
$\overset{!}{=}$ 

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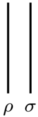
$\mathcal{D}' \cap \mathcal{C} = \Big\{ \rho \in \mathcal{C} : \varepsilon(\sigma, \rho)\varepsilon(\rho, \sigma) \equiv$  $\overset{!}{=}$  $\equiv 1_{\rho \otimes \sigma}$ for all $\sigma \in \mathcal{D} \Big\}$

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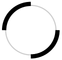
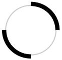
$\mu_{\mathcal{A}} = \Big[\mathcal{A}\big($  $\big)' : \mathcal{A}\big($  $\big)\Big] < \infty$

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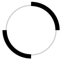
$\mu_{\mathcal{A}} = \Big[\mathcal{A}\Big($  $\Big)' : \mathcal{A}\Big($  $\Big)\Big] < \infty$

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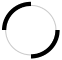
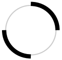
$\mu_{\mathcal{A}} = \Big[\mathcal{A}\Big($  $\Big)' : \mathcal{A}\Big($  $\Big)\Big] < \infty$

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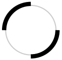
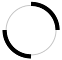
$\mu_{\mathcal{A}} = \Big[\mathcal{A}\Big($  $\Big)' : \mathcal{A}\Big($  $\Big)\Big] < \infty$

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$\rightsquigarrow$ **Reconstruction Program** [Jones]. Examples: [B. '15][Xu '16][Evans–Gannon]

Proposition [B. '16]

Let $N \subset M$ be a finite index finite depth subfactor. If there is a conformal net $\mathcal{A}$ such that $\mathrm{Rep}(\mathcal{A}) \cong Z(\mathcal{F}_{N\subset M})$, then $N \subset M$ *arises* from $\mathcal{A}$.

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Theorem ($\cdots$ - completely rational case [B. '17])

Let K be a proper hypergroup acting on a completely rational net $\mathcal{A}$.

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Interpretation (in analogy with finite groups):

- $\mathcal{F}$ is obtained by α^+-induction $\sim$ "K-twisted representations of $\mathcal{A}$"
- $\mathrm{Rep}(\mathcal{A}^K) =$ "$\mathcal{F}^K \sim K$-equivariantization"

Consider unitary fusion category $\mathcal{F}$ with Grothendieck ring $F = K_0(\mathcal{F}) = G \cup \{\rho\}$ with $\rho^2 = \sum_{g\in G} g$ and $\rho g = g\rho = \rho$.

- Then G is abelian and $\mathcal{F}$ are given by a non-degenerate bicharacter (Fourier transformation) on G and a sign (Frobenius–Schur indicator)
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Let $\mathcal{F}$ as above with G odd (abelian) group, then there is

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$\mathcal{F} \cong \mathrm{Hilb}_G \rtimes \mathbb{Z}_2$ is a nilpotent fusion category $\rightsquigarrow F/\!\!/G \cong \mathbb{Z}_2$
$\mathcal{A}_L^F(I) \subset \mathcal{A}_L(I) \sim R^{\mathbb{Z}_2} \subset R \rtimes \Delta(G)$ for action of $(G \rtimes_{-1} \mathbb{Z}_2) \times G$ on R
$\mathbb{Z}_2$-action on $A_{\tilde{L}}$ and choose $L = (\tilde{L} \times \tilde{L}') \oplus G$ and $\tilde{L}'$ mirror of $\tilde{L}$
Uses that (G, q) lifts always lifts to a lattice $\tilde{L}$ [Nikulin '79] relation to real projective K3 surfaces???

- **Izumi–Xu** unitary fusion categories $\mathcal{F}$, i.e. $F = K_0(\mathcal{F}) = G \cup \{\rho\}$ with $\rho^2 = |G|\rho + \sum_{g\in G} g$
- **Cuntz algebra approach** [Izumi '01][Evans–Gannon '13] $\rightsquigarrow$ polynomial eqs.
- Existence: solution only known for $|G| \leq 13$ (conjectured for all odd G)

Conjecture

For every G odd abelian group there exists an even lattice L and an action[a] of $K = F/\!\!/G = \{\mathrm{id}, \phi\}$ on $\mathcal{A}_L$, such that $\mathrm{Rep}(\mathcal{A}_L^F) = Z(\mathcal{F})$ for some categorification $\mathcal{F}$ of F.

[a] conj: involving lattice lifts of (G, q) and (G', q') with $|G'| = |G| + 4$

G	L	$\mathcal{A}_L^K$	
$\mathbb{Z}_1$	E_8	$\mathcal{A}_{G_{2,1}\times F_{4,1}}$	[Dynkin '52]* [?][B. '16]
$\mathbb{Z}_2$	D_8	$\mathcal{A}_{\mathrm{SU}(2)_{10}\times\mathrm{Spin}(11)_1} \rtimes \mathbb{Z}_2$	[B. '16]
$\mathbb{Z}_3$	$E_6 \times A_2$	$\mathcal{A}_{G_{2,3}\times\mathrm{SU}(2)_1}$	[Dynkin '52][Xu unpublished]
$\mathbb{Z}_3 \times \mathbb{Z}_3$	$(\mathrm{E}_6 \times \mathcal{A}_2)^2$	$\mathcal{A}_{\mathrm{Hg}} \otimes \mathcal{A}_{E_6\times A_2}$	hypothetical [EvGa '11]

*these come from inclusion of corresponding Lie algebras already studied by Dynkin

Thank you for your attention!