A planar algebraic description of defect lines in conformal field theory*

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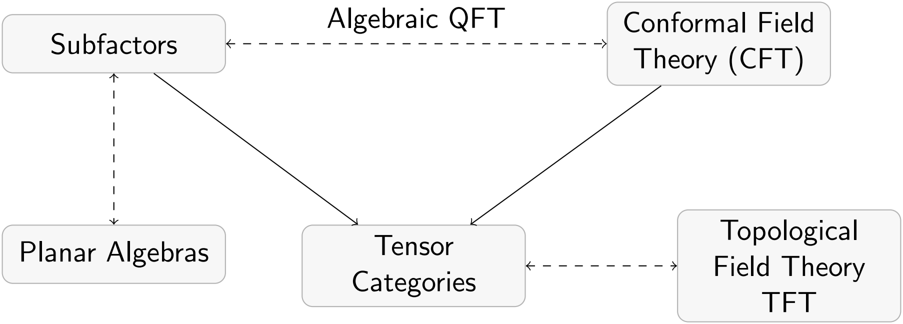
*based on work with R. Longo, Y. Kawahigashi and K.-H. Rehren
arXiv:1405.7863, arXiv:1407.4793, (see also: arXiv:1410.8848)



In binding together elements long-known but heretofore scattered and appearing unrelated to one another, it suddenly brings order where there reigned apparent chaos — Henri Poincaré

Question

Do all (finite index finite depth) subfactors arise from conformal field theory?

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Desired construction

Subfactor $N \subset M$	$\longmapsto$	Conformal net $\mathcal{A}_{N\subset M}$

such that $\mathrm{Rep}(\mathcal{A}_{N\subset M}) \cong D(N \subset M)$ ($=$ quantum double*)

> **"Royal road":** *[...] extract from the subfactor the Boltzmann weights of a critical two-dimensional lattice model then construct a quantum field theory from the scaling limit of the n-point functions.* (Jones 2015)

* $D(N \subset M) =$ unitary modular tensor category obtained from Ocneanu's asymptotic inclusion, Popa's SE algebra, Longo–Rehren subfactor, Drinfeld center

Definition (Conformal Net)

$S_1 \supset I \longmapsto \mathcal{A}(I) = \mathcal{A}(I)'' \subset \mathrm{B}(\mathcal{H})$, such that

1. Isotony: $I \subset J \Rightarrow \mathcal{A}(I) \subset \mathcal{A}(J)$
2. Locality: $[\mathcal{A}(I), \mathcal{A}(J)] = \{0\}$ if $I \cap J = \emptyset$.
3. Covariance: U is a unitary **positive-energy** representation of the Möbius group, s.t. $U(g)\mathcal{A}(I)U(g)^* = \mathcal{A}(gI)$.
4. Vacuum: $\exists\ \Omega$ is a (up to a phase) unique vector invariant under the Möbius group, s.t. $\overline{\vee_I \mathcal{A}(I)\Omega} = \mathcal{H}$.

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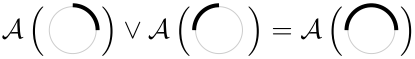
It follows that $\mathcal{A}(I)$ are type III_1 factors and Ω is cyclic and separating for $\mathcal{A}(I)$. Let

$N := \mathcal{A}\Big($  $\Big) \subset M := \mathcal{A}\Big($  $\Big) \supset \tilde{N} := \mathcal{A}\Big($  $\Big)$

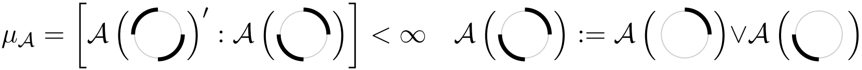
Then $(N \subset M, \tilde{N} \subset N' \cap M, \Omega)$ is a complete invariant and every such triple gives a conformal net (possibly reducible) provided Ω is cyclic for $N, \tilde{N}$ and separating for M and $\sigma^t_{(M,\Omega)}(N) \subset N$ for $t \leq 0$.

A conformal net $\mathcal{A}$ is **completely rational** if:

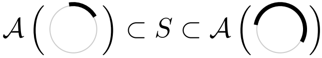
- Strong additivity. $N \vee \tilde{N} = M$:



  (Then $\tilde{N} = N' \cap M$ holds. $(N \subset M, \Omega)$ is a complete invariant.)
- Finite μ-index: finite Jones index of subfactor



- Split property. For every inclusion $\bar{I} \subset J$ of intervals $\exists$ intermediate **type** I **factor** S, e.g.:



  This holds if $\mathrm{Tr}(e^{-\beta L_0}) < \infty$ for all $\beta > 0$, where L_0 is the generator of rotations: $U(z \mapsto e^{\mathrm{i}t} z) = e^{\mathrm{i}t L_0}$.

Representation of $\mathcal{A} = \{\mathcal{A}(I)\}_{I\subset S^1}$ is a family:

$$\pi = \{\pi_I \colon \mathcal{A}(I) \to \mathrm{B}(\mathcal{H}_\pi)\}\,,$$

which is compatible, i.e. $\pi_J \restriction \mathcal{A}_0(I) = \pi_I$ for $I \subset J$.

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- Every π unitarily equivalent to a localized endomorphism $\rho \in \mathrm{End}(\mathcal{A}(I))$.
- Statistical dimension $d = [\mathcal{A}(I) : \rho(\mathcal{A}(I))]^{\frac{1}{2}}$.
- Tensor product: composition of localized endomorphisms.
- $\exists$ natural braiding $\{\varepsilon_{\rho,\sigma} \colon \rho \circ \sigma \to \sigma \circ \rho\}$ (Fredenhagen, Rehren, Schroer (1989)).

Theorem ((Kawahigashi, Longo, Müger (2001)))

Let $\mathcal{A}_0$ *be a* **completely rational** *conformal net. Then* $\mathrm{Rep}(\mathcal{A}_0)$ *is a* **modular C*-tensor category = unitary modular tensor category (UMTC)**.

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Example

Loop group net of $\mathrm{SU}(N)$ **at level** k**:** (see (Wassermann '98))

$$\mathcal{A}_{\mathrm{SU}(N),k}(I) = \pi(\mathrm{L}_I\mathrm{SU}(N))'' \qquad \text{(completely rational (Xu '00))}$$

with π level k vacuum PER of **loop group** $\mathrm{LSU}(N) = \mathrm{C}^\infty(S^1, \mathrm{SU}(N))$.

Extensions of nets are characterized by one **local** *extension*

We write if $\mathcal{B} \supset \mathcal{A}$ if

- $\mathcal{A}(J) \subset \mathcal{B}(J)$ for all $J \in S^1 \setminus \{-1\}$ finite index, irreducible.
- Relatively local extension:

$$[\mathcal{A}(I_1), \mathcal{B}(I_2)] = \{0\} \text{ if } I_1 \cap I_2 = \emptyset$$

Completely characterized by $M := \mathcal{B}(I) \supset N := \mathcal{A}(I)$ for a fixed I.

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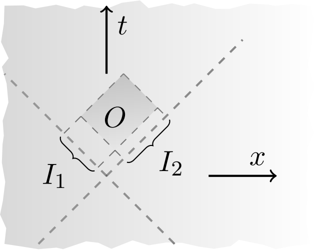
Theorem ((Longo–Rehren '95))

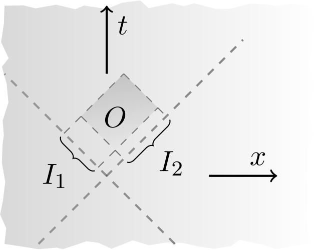
Conversely, every $M \supset N := \mathcal{A}(I)$, finite index, irreducible overfactor M, such that ${}_N M_N$ (actually the dual canonical endomorphism $\theta\colon N \to N$) is a representation of $\mathcal{A}$ localized in I.
$\mathcal{B}$ is a local net iff multiplication ${}_N M_N \otimes_N {}_N M_M \to {}_N M_N$ is commutative (with respect to the braiding of $\mathrm{Rep}(\mathcal{A})$).



One can define a **conformal net** on **Minkowski space** by

$$\mathcal{A}_2(O) = \mathcal{A}(I_1) \otimes \mathcal{A}(I_2)$$

where $\mathcal{A}_\pm$ are conformal nets on $\mathbb{R}$.



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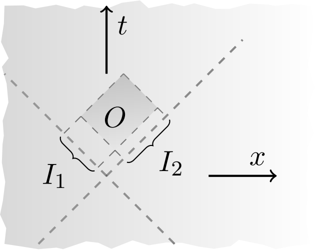
Full CFTs based on $\mathcal{A}$ are given by maximal local extensions

$$\mathcal{B}(O) \supset \mathcal{A}_2(O) \equiv \mathcal{A}(I_1) \otimes \mathcal{A}(I_2)\,.$$

Locality. $[\mathcal{B}(O_1), \mathcal{B}(O_2)] = \{0\}$ if O_1 and O_2 are space like separated.



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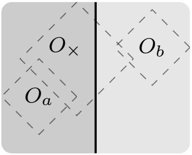
Theorem ((B–Kawahigashi–Longo '14))

There is a one-to-one correspondence between

- *Full CFTs* $\mathcal{B}_2 \supset \mathcal{A}_2$
- $M \supset N = \mathcal{A}(I)$ *with* ${}_N M_N \in \mathrm{Rep}(\mathcal{A})$ *up to Morita equivalence.*



$$\begin{Bmatrix} O_a \\ O_\times \\ O_b \end{Bmatrix} \longmapsto \begin{Bmatrix} \mathcal{B}_a(O_a) \\ \mathcal{D}(O_\times) \\ \mathcal{B}_b(O_b) \end{Bmatrix} \supset \mathcal{A}_2(O.)$$

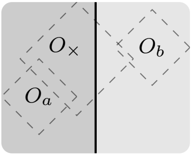
$$\begin{array}{ccccc} \mathcal{D}_+ & \subset & \mathcal{D} & \supset & \mathcal{D}_- \\ \cup & \swarrow & & \searrow & \cup \\ \mathcal{B}_a & & & & \mathcal{B}_b \\ & \searrow & & \swarrow & \\ & & \mathcal{A}_2 & & \end{array}$$

where $\mathcal{D}_\pm$ left/right center:

$$\mathcal{A}_2 \subset \mathcal{D}_\pm \subset \mathcal{D}$$

maximal intermediate local nets.

Two different full CFTs $\mathcal{B}_a, \mathcal{B}_b \supset \mathcal{A}_2$ divided by a defect line



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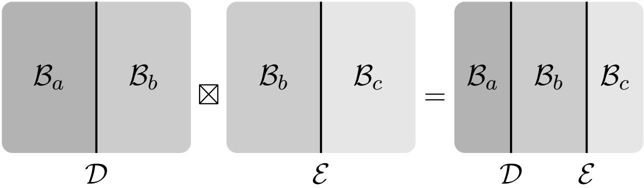
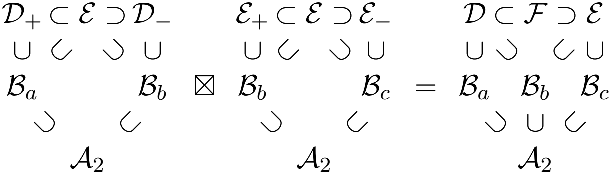
maximal intermediate local nets.

- Defect line invisible for the subnet $\mathcal{A}_2$ (conserves symmetries prescribed by $\mathcal{A}$)
- Different realization $\leftrightarrow$ different boundary conditions
- $\mathcal{A}$-topological $\mathcal{B}_a$-$\mathcal{B}_b$ defect line.

Fusion of defect lines

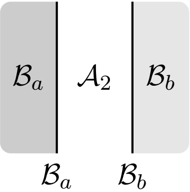




$\mathcal{F} = \mathcal{D} \otimes_{\mathcal{B}_a^<} \mathcal{E}$ can be defined Connes' fusion over wedge algebra or using the braiding.

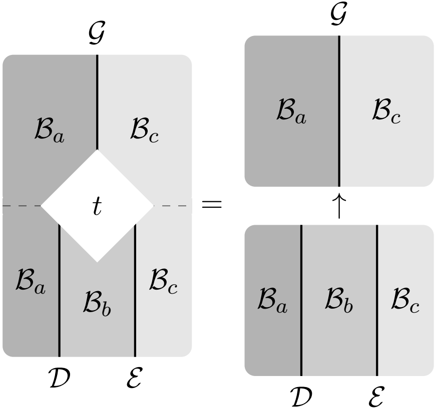
Fuse trivial defects over $\mathcal{A}_2$ (not a full CFT!!) $\rightsquigarrow$ non-factorial "defect line"

$$\begin{array}{ccc} & \mathcal{B}_a & \\ \rotatebox{45}{$\subset$} & & \rotatebox{-45}{$\supset$} \\ \mathcal{B}_a & & \mathcal{A}_2 \\ \rotatebox{-45}{$\supset$} & & \rotatebox{45}{$\subset$} \\ & \mathcal{A}_2 & \end{array} \boxtimes \begin{array}{ccc} & \mathcal{B}_b & \\ \rotatebox{45}{$\subset$} & & \rotatebox{-45}{$\supset$} \\ \mathcal{A}_2 & & \mathcal{B}_b \\ \rotatebox{-45}{$\supset$} & & \rotatebox{45}{$\subset$} \\ & \mathcal{A}_2 & \end{array} = \begin{array}{ccccc} \mathcal{D}^{\mathrm{u}}_{+} & \subset & \mathcal{D}^{\mathrm{u}} & \supset & \mathcal{D}^{\mathrm{u}}_{-} \\ \cup & \rotatebox{45}{$\subset$} & & \rotatebox{-45}{$\supset$} & \cup \\ \mathcal{B}_a & & & & \mathcal{B}_b \\ & \rotatebox{-45}{$\supset$} & & \rotatebox{45}{$\subset$} & \\ & & \mathcal{A}_2 & & \end{array}$$



Theorem ((B–Kawahigashi–Longo–Rehren '14))

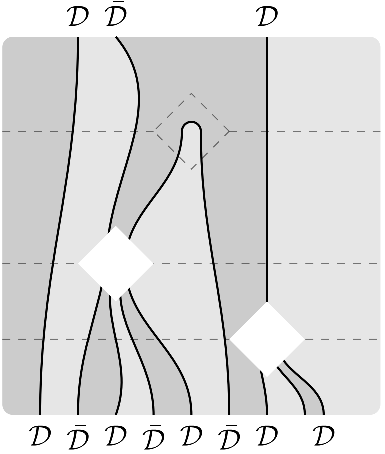
1. $\mathcal{D}^{\mathrm{u}}(O)' \cap \mathcal{D}^{\mathrm{u}}(O) = \mathcal{A}_2(O)' \cap \mathcal{D}^{\mathrm{u}}(O)$, *i.e.* $\mathcal{D}^{\mathrm{u}}(O)$ *has finite center.*
2. *Every minimal central projection* $p \in \mathcal{D}(O)$ *yields an irreducible* $\mathcal{A}$*-topological* $\mathcal{B}_a$–$\mathcal{B}_b$*-defect* $\mathcal{D}_p \cong \mathcal{D}p$.
3. *Every irreducible* $\mathcal{A}$*-topological* $\mathcal{B}_a$–$\mathcal{B}_b$*-defect arises this way.*
4. *Minimal projections* $\stackrel{1:1}{\longleftrightarrow}$ *irreducible* M_a-M_b *sectors related to* $\mathrm{Rep}(\mathcal{A})$.



$$
\begin{array}{ccccccc}
\mathcal{G}_+ & \subset & \mathcal{G} & \supset & \mathcal{G}_- \\
\cup & & & & \cup \\
\mathcal{B}_a & & & & \mathcal{B}_c \\
& & \mathcal{A}_2 & & \\
& & \uparrow & & \\
\mathcal{D} & \subset & \mathcal{F} & \supset & \mathcal{E} \\
\cup & & & & \cup \\
\mathcal{B}_a & & \mathcal{B}_b & & \mathcal{B}_c \\
& & \cup & & \\
& & \mathcal{A}_2 & &
\end{array}
$$

Bounded maps $t\colon \mathcal{H}_\mathcal{F} \to \mathcal{H}_\mathcal{G}$, which are $\mathcal{B}_a$,$\mathcal{B}_b$ equivariant, i.e. $ta = at$ for $\in \mathcal{B}_a, \mathcal{B}_b$.

Fix $\mathcal{D}$ and $\mathcal{A}$-topological $\mathcal{B}_a$–$\mathcal{B}_b$-defect and $\bar{\mathcal{D}}$ the dual $\mathcal{B}_b$–$\mathcal{B}_a$-defect.



$\rightsquigarrow$ action of planar operad on spaces of intertwiners

Theorem (B. unpublished)

On obtains a subfactor planar algebra. This is the planar algebra of the subfactor related to $\mathrm{Rep}(\mathcal{A})$ which characterizes $\mathcal{D}$.

- $N := \mathcal{A}(I)$, ${}_N\mathcal{C}_N := \mathrm{Rep}^I(\mathcal{A})$, then $\mathcal{B}_\bullet \supset \mathcal{A}_2$ are characterized by subfactors $M_\bullet \supset N$ with ${}_N M_{\bullet N} \in {}_N\mathcal{C}_N$. $\mathcal{D}$ is characterized by a sector $\beta\colon M_a \to M_b \in {}_{M_b}\mathcal{C}_{M_a}$.
- $\mathcal{A}_2(O) \subset \mathcal{B}_\bullet(O)$ is conjugated to a Longo–Rehren subfactor and yields a Morita equivalence:

$$ {}_N\mathcal{C}_N \boxtimes {}_N\mathcal{C}_N^{\mathrm{op}} \sim {}_{M_\bullet}\mathcal{C}_{M_\bullet} \boxtimes {}_{M_\bullet}\mathcal{C}_{M_\bullet}^{\mathrm{op}} $$

- $\mathcal{B}_a(O) \subset \mathcal{D}(O)$ yields a Morita equivalence:

$$ {}_{M_a}\mathcal{C}_{M_a} \boxtimes {}_{M_a}\mathcal{C}_{M_a}^{\mathrm{op}} \sim {}_{M_b}\mathcal{C}_{M_b} \boxtimes {}_{M_a}\mathcal{C}_{M_a}^{\mathrm{op}} $$

- $\mathcal{B}_a(O) \subset \mathcal{D}(O)$ is actually conjugated to $\beta(M_a) \subset M_b$.
- The inclusion

$$ \mathcal{B}_a(O) \subset \mathcal{D}(O) \subset (\bar{\mathcal{D}} \boxtimes \mathcal{D})(O) \subset (\mathcal{D} \boxtimes \bar{\mathcal{D}} \boxtimes \mathcal{D} \subset)(O) \subset \cdots $$

is essentially the Jones tower and every element in the relative commutant is already a defect intertwiner (full inclusion) and vice versa.

$[M:N] < 4$ (Classified by Jones, Ocneanu, ...)

- A_k, D_{2n}, E_6, E_8 (in pairs): All arise in the corresponding ADE classification of $\mathrm{SU}(2)$ CFTs of Cappelli, Itzykson and Zuber (1987), observed by Ocneanu.

$[M:N] = 4$ (Classified by Popa)

- Affine Dynkin diagrams: ?

$4 < [M:N] < 5$

- GHJ: related to E_6 above
- 2221: $G_{2,3} \subset E_{6,1}$
- Haagerup: Conjectured by Evans-Gannon.
- Aseda–Haagerup: ?
- Extended Haagerup: ?

$[M:N] > 5$ ...

Let $N := \mathcal{A}(I)$, ${}_N\mathcal{C}_N := \mathrm{Rep}^I(\mathcal{A})$. There is a functor from the

- 2-category of
  Morita classes of subfactors $[N \subset M_a]$ based on ${}_N\mathcal{C}_N$, Morphisms $\beta\colon M_a \to M_b$ based on ${}_N\mathcal{C}_N$ and intertwiner $t \in \mathrm{Hom}(\beta_1, \beta_2)$
- 2-category of
  full CFTs based on $\mathcal{A}$, $\mathcal{A}$-topological defects and interwiners.

which is an equivalence.
Higher structure of quantum double/center which maps ${}_N\mathcal{C}_N \mapsto Z({}_N\mathcal{C}_N)$.

First quantization is a mystery, but second quantization is a functor! – Edward Nelson

Theorem

Let $\mathcal{A}$ be a conformal net with $\mathrm{Rep}(\mathcal{A}) = D(N \subset M)$, then the planar algebra of $N \subset M$ prescribes a certain topological defect line of full CFTs based on $\mathcal{A}_{N\subset M}$.

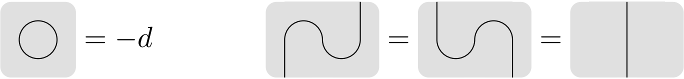
Desired construction

Subfactor $N \subset M$	$\longmapsto$	Conformal net $\mathcal{A}_{N\subset M}$

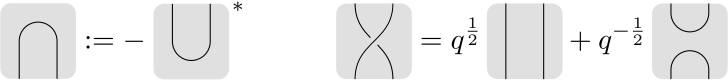
such that $\mathrm{Rep}(\mathcal{A}_{N\subset M}) \cong D(N \subset M)$ ($=$ quantum double)

Thank you!

Example $G = \mathrm{SU}(2)$**:** Irreducible representations $\{0, \frac{1}{2}, 1, \ldots, \frac{k}{2}\}$.
$\mathrm{Rep}(\mathcal{A}_{\mathrm{SU}(2),k})$ is generated by $\frac{1}{2}$-representation ρ and $\cup \in \mathrm{Hom}(\mathrm{id}, \rho\rho)$:



with $\cap \in \mathrm{Hom}(\rho\rho, \mathrm{id})$ and **braiding** defined by **Kaufmann bracket**



where $q = e^{\frac{\mathrm{i}\pi}{k+2}}$, $d = q + q^{-1} = 2\cos\left(\frac{\pi}{k+2}\right)$.